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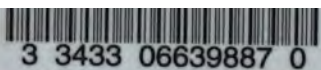
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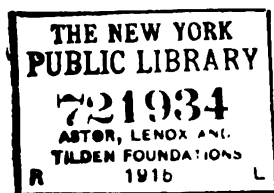
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PREFACE.

The first two of the five volumes of this set contain all the Instruction and Question Papers used by the students of our Course in Refrigeration. The Papers on Pneumatics and Heat give the information necessary to the proper study of the theory of refrigeration, and the Paper on Steam and Steam Engines not only affords the means of applying the principles given in the two preceding Papers, but also gives information that will prove of value to all who have anything to do with ice-making and refrigerating machinery.

The two Papers entitled "Principles of Refrigeration" and "Refrigerating and Ice-Making Machinery" form the only work on this subject that enters into the minute details of practical refrigeration, of the various types of machines,—both compression and absorption,—and at the same time gives sufficient of the theory to fully explain and substantiate any statement that may be made therein. These Papers give the practical engineer enough of the theory to enable him to handle the problems that he may meet with in his work, and explain in simple language why a refrigerating or ice-making machine does its work; they give him those practical pointers regarding the operating of a plant that will enable him to obtain economically the maximum refrigerating effect, and show him how to tell from general observation the capacity and efficiency of his plant and how it is running. These Papers also contain considerable information relating to the erection, operation, and installation of plants, and give general rules for remodeling an old plant or designing a new one.

The volumes of this Course are five in number, and are made up as follows:

Vol. I contains the Instruction and Question Papers on Arithmetic, Mensuration, Elementary Algebra and Trigonometric Functions, Logarithms, and Elementary Mechanics.

Vol. II contains the Instruction and Question Papers on Pneumatics, Heat, Steam and Steam Engines, Principles of Refrigeration, and Refrigerating and Ice-Making Machinery.

Vol. III contains the Drawing Plates and the instructions for drawing them. It forms a very complete course in Mechanical Drawing.

Vol. IV contains the tables and formulas given in the various Instruction Papers. The student that has finished his Course will find this volume of special value. The tables and also the formulas, with the definitions of the letters used in them, are conveniently arranged for reference, so that the student is saved the time and labor of searching for them in the Instruction Papers.

Vol. V contains the answers to the questions and the solutions to the examples in the Question Papers.

The Papers on Pneumatics, Heat, and Steam and Steam Engines have been printed from the same plates that were used in printing the same Papers in the bound volumes of our Complete Mechanical Course; hence, there is a break in the page, article, figure, etc., numbers between Vol. I and Vol. II, and a slight break in the page numbers between the Papers on Heat and Steam and Steam Engines; this does not affect the subject matter, however, which is published in full and in regular order.

THE INTERNATIONAL CORRESPONDENCE SCHOOLS.

HOY WAY
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CONTENTS.

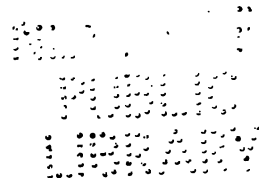
ARITHMETIC.	PAGE
Definitions, - - - - -	1
Notation and Numeration, - - - - -	1
Addition, - - - - -	4
Subtraction, - - - - -	9
Multiplication, - - - - -	12
Division, - - - - -	17
Cancellation, - - - - -	21
Fractions, - - - - -	23
Decimals, - - - - -	38
Percentage, - - - - -	55
Denominate Numbers, - - - - -	61
Involution, - - - - -	76
Evolution, - - - - -	79
Ratio, - - - - -	96
Proportion, - - - - -	100
MENSURATION AND USE OF LETTERS IN ALGEBRAIC FORMULAS.	
Formulas, - - - - -	115
Lines and Angles, - - - - -	119
Quadrilaterals, - - - - -	123
The Triangle, - - - - -	126
Polygons, - - - - -	130
The Circle, - - - - -	133
The Prism and Cylinder, - - - - -	138
The Pyramid and Cone, - - - - -	142
The Frustum of a Pyramid or Cone, - - - - -	143
The Sphere and Cylindrical Ring, - - - - -	145

ELEMENTARY ALGEBRA AND TRIGONOMETRIC FUNCTIONS.							PAGE
Use of Letters,	-	-	-	-	-	-	149
Notation,	-	-	-	-	-	-	151
Reading Algebraic Expressions,	-	-	-	-	-	-	154
Positive and Negative Quantities,	-	-	-	-	-	-	155
Addition and Subtraction,	-	-	-	-	-	-	158
Symbols of Aggregation,	-	-	-	-	-	-	164
Multiplication,	-	-	-	-	-	-	166
Division,	-	-	-	-	-	-	172
Factoring,	-	-	-	-	-	-	175
Fractions,	-	-	-	-	-	-	181
Theory of Exponents,	-	-	-	-	-	-	194
Equations,	-	-	-	-	-	-	198
Trigonometric Functions,	-	-	-	-	-	-	208
Trigonometric Tables,	-	-	-	-	-	-	210
Solution of Triangles,	-	-	-	-	-	-	215
LOGARITHMS.							
Exponents,	-	-	-	-	-	-	221
The Logarithmic Table,	-	-	-	-	-	-	224
Multiplication by Logarithms,	-	-	-	-	-	-	231
Division by Logarithms,	-	-	-	-	-	-	232
Involution by Logarithms,	-	-	-	-	-	-	235
Evolution by Logarithms,	-	-	-	-	-	-	237
ELEMENTARY MECHANICS.							
Matter and its Properties,	-	-	-	-	-	-	241
Motion and Rest,	-	-	-	-	-	-	245
Force,	-	-	-	-	-	-	247
Three Laws of Motion,	-	-	-	-	-	-	247
Composition of Forces,	-	-	-	-	-	-	251
Resolution of Forces,	-	-	-	-	-	-	259
Dynamics,	-	-	-	-	-	-	261
Gravitation,	-	-	-	-	-	-	261
Falling Bodies,	-	-	-	-	-	-	264
Projectiles,	-	-	-	-	-	-	269

CONTENTS.

vii

ELEMENTARY MECHANICS— <i>Continued.</i>		PAGE
Centrifugal Force, - - - - -		271
Statics, - - - - -		273
Moments of Forces, - - - - -		273
Center of Gravity, - - - - -		274
Equilibrium, - - - - -		278
Simple Machines, - - - - -		279
Friction, - - - - -		292
Efficiency, - - - - -		295
Work and Energy, - - - - -		296
Density and Specific Gravity, - - - - -		302
QUESTIONS AND EXAMPLES.		PAGE
Arithmetic, - - - - - Questions 1-76		307
Arithmetic, - - - - - Questions 77-141		315
Mensuration and Use of Letters in Algebraic Formulas, Questions 142-213		323
Elementary Algebra and Trigonometric Functions, - Questions 214-268		331
Logarithms, - - - - - Questions 269-283		339
Elementary Mechanics, - Questions 355-453		341



ROY WEN
CLUB
YASRU

INDEX.

A.		PAGE			PAGE
Abstract number		1	Area of parallelogram		124
Accelerating force		265	" " prism		139
Acceleration, Definition of		265	" " pyramid		142
Acute angle		120	" " regular polygon		131
Addition in algebra		158	" " sector of circle		137
" of decimals		40	" " segment of circle		137
" " denominate numbers		63	" " sphere		145
" " fractions	29, 186		" " trapezoid		124
" " monomials		160	" " triangle		129
" " polynomials		163	Atom		241
" Proof of		8	Avoirdupois weight		63
" Rule for		8			
" Sign of		4	B.		PAGE
" table		5	Base (in percentage)		56
Adjacent angles		120	Base of logarithmic system		221
Aeriform body		242	Bent lever, Principle of		281
Aggregation, Symbols of	53, 164		Binomial, Definition of		153
Algebra, Notation used in		151	Block		285
" Signs used in		151	Blow, Force of		301
" Use of letters in		149	Body, Aeriform		242
Algebraic expressions, Reading		154	" Gaseous		242
" " Terms of		153	" Liquid		242
Altitude of cone		142	" Solid		242
" " frustum of cone or py-			Brace		53
amid		144	Brackets		53
" " parallelogram		123	Briggs' logarithms		221
" " pyramid		142	Brittleness		244
" " triangle		129	Bushel, Cubic inches in		65
Amount (in percentage)		56			
Angle		120	C.		PAGE
" Acute		120	Cancellation		21
" Complement of		209	" in transforming equa-		
" Functions of		209	tion		200
" Inscribed		133	" Rule for		23
" Obtuse		120	Capacity, Measures of		64
" Right		120	Cause and effect		108
" Supplement of		209	Center of gravity		274
Angles, Adjacent		120	" " " of irregular		
Antecedent (of a ratio)		97	figure		277
Arc of circle		133	" " " of parallelogram		276
Area of circle		136	" " " solid		277
" " cone		142	" " " triangle		276
" " cylinder		139	" " moments		273
" " cylindrical ring		146	Centrifugal force		271
" " frustum of cone		144	Centripetal force		272
" " " pyramid		144			

	PAGE		PAGE
Characteristic of logarithm	222	Cylinder, Convex surface of	139
" " " Rules	223	" Volume of	139
" " " for	223	Cylindrical ring	146
Chord of circle	131	" " Convex area of	146
Cipher	2	" " Volume of	147
Circle	120		
" Arc of	133	D.	PAGE
" Area of	136	" parts of a foot, To reduce	38
" Center of	120	inches to	50
" Chord of	133	" point	38
" Circumference of	120	" To express as fraction hav-	
" Diameter of	133	ing given denominator	52
" Radius of	133	" To reduce a fraction to	50
" Sector of	136	" " to a fraction	51
" Segment of	137	Decimals, Addition of	40
Circular ring	136	" Division of	45
Circumference	120	" Multiplication of	43
Clearing equation of fractions	200	" Subtraction of	42
Coefficient, Definition of	152	Denominate number	61
" of friction	292	" " Compound	62
Combination of pulleys	286	" " Simple	61
Common denominator	209	Denominate numbers, Addition of	68
Complement of an angle	209	" " Division of	73
Complex fraction	192	" " Multiplication	72
Components of a force	260	" " of	72
Composite numbers	21	" " Reduction of,	
Composition of forces	251	to higher	
" " matter	241	denomina-	
Compound denominate number	62	tions	67
" lever, Principle of	281	" " Reduction of,	
Compressibility	244	to lower de-	
Concrete number	1	nomina-	
Cone	142	tions	65
" Altitude of	142	" " Subtrac-	
" Convex area of	142	tion of	70
" Frustum of	143	Denominator	23
" Slant height of	142	" Common	27
" Volume of	142	" Least common	27
Consequent (of a ratio)	97	Density, Definition of	302
Conservation of energy	301	Diameter of circle	133
Convex surface of cone	142	Difference	9
" " " cylindrical ring	146	" (in percentage)	56
" " " pyramid	142	Differential pulley	287
" " " solid	139	Digit	2
Cosine of angle	209	Direct proportion	101
Cotangent of angle	210	" ratio	97
Couple, Definition of	273	Dividend	17
Couplet (ratio and proportion)	97	Divisibility	243
Cube	77	Division	17
" of a number	77	" by logarithms	232
" root	79, 86	" in algebra	172
" " Proof of	92	" of decimals	45
" " Rule for	92	" " denominate numbers	73
Cubic measure	63	" " fractions	34, 189
Curved line	119	" " monomials	173
Cylinder	138		

	PAGE		PAGE
Division of polynomials	173	Foot-pound	296
" Proof of	20	Force, Accelerating	265
" Rule for	20	" Centrifugal	271
" Sign of	17	" Centripetal	272
Divisor	17	" Components of	260
Driver (wheelwork)	284	" Definition of	247
Dry measure	64	" Moment of	273
Ductility	245	" of a blow	301
Dynamics, Definition of	248, 261	" " gravity	262
E.		" Representation of, by a line	251
Efficiency, Definition of	295	" Retarding	265
Elasticity	244	" To resolve, into components	260
Energy, Conservation of	301	" Unit of	248
" Definition of	298	Forces, Composition of	251
" Kinetic	298	" Parallelogram of	253
" Potential	300	" Polygon of	256
Equality, Sign of	4	" Resolution of	259
Equation, Clearing, of fractions	200	" Resultant of	252
" Definition of	150	" Triangle of	255
" how solved	198	Formulas	115
" Literal	203	" for falling bodies	267
" Members of	198	" " gravity problems	263
Equations, Simple, Problems lead-		" how applied	117
ing to	204	" Symbols used in	115
" " Solution of	201	Fraction	23
" " with one un-		" Complex	192
known quan-		" Improper	25
tity	201	" Proper	25
" Transformations of	199	" Terms of	25
Equilateral triangle	126	" Three signs of	181
Equilibrium	248, 278	" To invert	35
" Neutral	278	" To reduce a decimal to	51
" Stable	278	" To reduce a whole or	
" Static	273	mixed number to	26
" Unstable	278	" To reduce to a decimal	50
Evolution	79	" To reduce to lowest terms	26
" by logarithms	237	" Value of	24
Expansibility	244	Fractional exponent	194
Exponent	77	Fractions, Addition and subtrac-	
" Definition of	152	tion of	186
" Fractional	194	" Addition of	29
" Literal	194	" Division of	34, 189
" Negative	194	" in algebra	181
Exponents, Theory of	194	" Least common denomi-	
Extension	243	nator of	27
" Measures of	62	" Multiplication of	32, 188
Extremes of a proportion	101	" Reduction of	25, 182
F.		" Reduction of, to common	
Factor, Prime	21	denominator	28
Factoring	175	" Roots of	94
Factors	21, 152	" Subtraction of	30
Falling bodies	264	Friction, Coefficient of	292
" " Formulas for	267	" Definition of	292
Follower (wheelwork)	284	" Laws of	293
		Frustum of cone, Convex surface of	144
		" " " or pyramid	143

	PAGE		PAGE
Frustum of cone, Volume of	144	Law of gravitation	262
Fulcrum of lever	279	" " motion, Newton's	248
Functions, Trigonometric	259	" " weight	262
G.		Letters, Use of, in algebra	249
	PAGE	Lever	279
Gain or loss per cent.	60	" Bent	281
Gallon, Cubic inches in	65	" Compound, Principle of	282
" Weight of	65	" Principle of	280
Gas, Permanent	242	" Weight arm and power arm	
Gaseous body	242	of	280
Gravitation, Definition of	261	" Weight, power, and fulcrum	
" Law of	262	of	279
Gravity, Center of	274	Like number	1
" Force of	262	" terms	153
" problems, Formulas for	263	Line, Curved	119
" Specific	302	" Horizontal	119
H.		" of direction	278
	PAGE	" perpendicular to another	119
Hardness	244	" Vertical	119
Heptagon	130	Linear measure	62
Hexagon	130	Lines, Parallel	119
Horizontal line	119	Liquid body	242
Horsepower, Definition of	208	" measure	64
Hydraulics	248	Literal equation	203
Hydrodynamics	248	" exponent	194
Hydrokinetics, Definition of	248	Logarithm, Characteristic and man-	
Hydrostatics, Definition of	249	tissa of	222
Hypotenuse	126	" Definition of	221
I.		Logarithmic table	224
	PAGE	Logarithms, Briggs' or common	221
Impenetrability	243	" Division by	232
Improper fractions	25	" Evolution by	237
Inches, To reduce, to decimal parts		" Hyperbolic, Napierian,	
of a foot	50	or natural	221
Inclined plane, Principle of	288	" Involution by	235
Indestructibility	244	" Multiplication of	231
Index of a root	79	" Process of finding, from	
Inertia	243	table	224
" Explanation of	249	Long ton table	63
Initial velocity	268	M.	
Inscribed angle	133		PAGE
" polygon	134	Machine, Efficiency of	295
Integer	2	Machines, Simple	279
Integral expression	190	Malleability	244
Inverse proportion	101	Mantissa of logarithm	222
" ratio	97	Mass of a body	262
Involution	76	Matter, Composition of	241
" by logarithms	235	" Definition of	241
Isosceles triangle	126	" General properties of	242
K.		" Special properties of	243
	PAGE	Means (of a proportion)	101
Kinetic energy	298	Measure, Cubic	63
Kinetics, Definition of	248	" Dry	64
Known quantity	198	" Linear	62
L.		" Liquid	64
	PAGE	" of money	65
Law of combination of pulleys	287		
" " friction	293		

INDEX.

xiii

	PAGE		PAGE
Measure of time	64	Numbers, Like	1
" Square	62	" Mixed	25
Measures of capacity	64	" Prime	21
" extension	62	" Unlike	1
" weight	63	Numeration	1, 4
Mechanics, Definition of	245	Numerator	23
Members of an equation	198	O.	PAGE
Mensuration, Definition of	119	Oblique-angled triangle, Solution	
Minuend	9	of	218
Minus	9	Obtuse angle	120
Mixed number	25	P.	PAGE
" quantity	191	Parabola, the path of a projectile	270
Mobility	243	Parallel lines	119
Molecule	241	Parallelogram	123
Moment of a force	273	" Altitude of	123
Moments, Center of	273	" Area of	124
Money, Measure of	65	" Center of gravity of	276
Monomial, Definition of	153	" Diagonal of	129
" To extract root of	197	" of forces	253
Monomials, Addition of	160	Parallelopipedon	138
" Division of	173	Parenthesis	53
" Multiplication of	168	" Removal of	165
" Subtraction of	160	Path of a moving body	245
Motion and rest, Explanation of	245	Pentagon	130
" Newton's laws of	248	Per cent.	55
Multiplicand	12	" Sign of	55
Multiplication	12	Percentage	55, 56
" in algebra	166	Perimeter of polygon	130
" decimals	43	Permanent gas	242
" denominate num- bers	72	Perpendicular, Meaning of	119
" fractions	32, 188	Pinion	284
" logarithms	231	Pitch of screw	201
" monomials	168	Plane figure	123
" polynomials	169	" Inclined	288
" Proof of	16	" trigonometry	208
" Rule for	16	Pneumatics, Definition of	248
" Sign of	12	Polygon	130
" table	13	" Area of	131
Multiplier	12	" Inscribed	134
N.	PAGE	" of forces	256
Napierian logarithms	221	" Perimeter of	130
Natural logarithms	221	" Regular	130
Negative exponent, Meaning of	194	" Sides of	130
" quantities, Explanation of	155	" Sum of interior angles of	131
Neutral equilibrium	278	Polynomial, Arrangement of, in in- creasing or decreas-	
Newton's laws of motion	248	ing powers	154
Notation	1, 4	" Definition of	154
" Arabic	2	Polynomials, Addition and subtrac-	
" in algebra	151	tion of	163
Numbers	1	" Division of	173
" Abstract	1	" Multiplication of	169
" Composite	21	Porosity	242
" Concrete	1		
" Denominate	61		

	PAGE		PAGE
Positive and negative quantities,		Quantities, Unlike	162
Explanation of	155	Quantity, Definition of	151
Potential energy	300	" Mixed	191
Pound, the unit of force	248	Quotient	17
Power arm of lever	280		
" Definition of	152	R.	PAGE
" Meaning of, in mechanics	207	Radical sign	79, 153
" of a number	76	Radius of circle	133
" Unit of	297	Range of a projectile	269
Powers and roots (in proportion)	106	Rate (in percentage)	56
Prime factor	21	Ratio	96
" number	21	" Couplet of a	97
Prism	138	" Direct	97
" Convex surface of	139	" how expressed	96
" Volume of	139	" Inverse	97
Problem, Statement of	204	" Terms of	97
Problems leading to simple equations	204	" Value of	97
Product	12	" Velocity	292
Projectile, Definition of	269	Reciprocal of a number	97
" Range of	269	" " quantity	181
Proof of addition	8	" ratio	97
" division	20	Rectangle	123
" multiplication	16	Reduction of denominate numbers	65
" subtraction	11	" " fractions	25, 182
Proper fraction	25	Regular polygon	130
Properties of matter, General	242	Remainder	9
" " " Special	243	Resolution of forces	259
Proportion	100	Resultant of three or more forces, how found	255
" Compound	109	" " forces, Definition of	252
" Couplet of a	100	Rhomboid	123
" Direct	101	Rhombus	123
" Extremes of a	101	Right angle	120
" Inverse	101, 104	" angled triangle	126
" Means of a	101	" " " Solution of	215
" Powers and roots in	106	Ring, Cylindrical	146
" Rules for	101	Root	77
" Simple	100	" Cube	86
Pulley	285	" Definition of	153
" Differential	287	" Index of	79
" Fixed	285	" Square	79
" Movable	285	Roots of fractions	94
Pulleys, Combination of	286	" other than square and cube	95
Pyramid	142		
" Altitude of	142	S.	PAGE
" Convex area of	142	Scalene triangle	126
" Frustum of	143	Screw	291
" Slant height of	142	Sector, Area of	137
" Vertex of	142	Segment, Area of	137
" Volume of	142	Semicircle	134
		Semicircumference	134
Q.	PAGE	Sign of addition	4
Quadrant	64	" division	17
Quadrilateral	123	" equality	4
Quantities, Known and unknown	198	" multiplication	12
" Positive and negative	155	" per cent.	55

XV

	PAGE		PAGE
Sign of subtraction	9	Terms of an algebraic expression	153
" Radical	79, 153	" " equation, Changing signs of .	200
Signs of terms, how changed	200	" " " Transposition of .	199
" used in algebra	151	" " a ratio	97
Simple denominate numbers	61	Thermodynamics, Definition of .	248
" equation, Solution of	201	Thread of screw	201
" " with one unknown quantity	201	Time, Measure of	64
" equations, Problems leading to	204	" Unit of	247
" machines	279	Train (wheelwork)	284
" proportion	109	Transformations of an equation .	199
Sine of angle	209	Transposition of terms of an equation	199
Slant height of cone or pyramid .	142	Trapezoid	123
Solid body	138, 150, 242	" Area of	124
Solution of right-angled triangle .	215	Triangle	126
" triangles	208, 215	" Altitude of	129
Space, Unit of	247	" Area of	129
Specific gravity, Definition of .	302	" Center of gravity of	276
Sphere	145	" Equilateral	126
" Area of surface of	145	" Hypotenuse of	126
" Volume of	146	" Isosceles	126
Square	77, 123	" Oblique-angled, Solution of	218
" foot	124	" of forces	255
" inch	124	" Relation between sides and angles of	215
" measure	62	" Right-angled	126
" root	79	" " Solution of	215
" " Proof of	84	" Scalene	126
" Rule for	84	" Solution of	208, 215
Stable equilibrium	278	Trigonometric functions	209
Statement of problem	204	" " how found from tables .	210
Static equilibrium	273	" " Relations between tables, Use of .	210
Statics, Definition of	248, 273	Trigonometry, Definition of	208
Subtraction	9	Trinomial, Definition of	153
" of decimals	42	" To factor	178
" " denominate numbers	70	Troy weight	63
" " fractions	30, 186		
" " monomials	160		
" " polynomials	163		
" Proof of	11		
" Rule for	11		
" Sign of	9		
Subtrahend	9		
Supplement of angle	209		
Surface of solid	139		
Symbols of aggregation	53, 164		
T.	PAGE	U.	PAGE
Table, Addition	5	Uniform velocity	245
" Multiplication	13	Unit	1
Tables, Trigonometric	210	" of a number	1
Tangent of angle	209	" force	248
Tenacity	244	" power	297
Terms, Like and unlike	153	" work	296
" of a fraction	25	" square	124
		United States money	65
		Units of velocity, space, and time .	246, 247
		Unknown quantity	198
		Unlike number	198

	PAGE		PAGE
Unlike quantities	162	Volume of frustum of cone or pyr-	
" terms	153	amid	144
Unstable equilibrium	278	" " prism	139
V.	PAGE	" " pyramid	142
Value of a fraction	24	" " sphere	146
" " " ratio	97	" Unit of	139
Vapor, Definition of	242	W.	PAGE
Variable velocity	245	Water, Gallons in a cubic foot of	65
Velocity, Definition of	245	Wedge	291
" Initial	268	Weight	243
" ratio	292	" arm of lever	280
" space, and time, Formulas		" Avoirdupois	63
connecting	246	" Laws of	262
" Uniform and variable	245	" Measures of	63
" Unit of	246	" Troy	63
Vertex of cone or pyramid	142	Wheel and axle	283
Vertical line	119	Wheelwork	284
Vinculum	53	Work, Definition of	296
Volume of cone	142	" Unit of	296
" " cylinder	139	Z.	PAGE
" " cylindrical ring	147	Zero	

ARITHMETIC.

DEFINITIONS.

1. Arithmetic is the art of reckoning, or the study of numbers.

2. A unit is *one*, or a single thing, as *one*, *one* bolt, *one* pulley, *one* dozen.

3. A number is a unit or a collection of units, as *one*, *three* engines, *five* boilers.

4. The unit of a number is one of the collection of units which constitutes the number. Thus, the unit of *twelve* is *one*, of *twenty* dollars is *one* dollar, of *one hundred* bolts is *one* bolt.

5. A concrete number is a number applied to some particular kind of object or quantity, as three *grate bars*, *five dollars*, ten *pounds*.

6. An abstract number is a number that is not applied to any object or quantity, as *three*, *five*, *ten*.

7. Like numbers are numbers which express units of the *same kind*, as 6 *days* and 10 *days*, 2 *feet* and 5 *feet*.

8. Unlike numbers are numbers which express units of *different kinds*, as ten *months* and eight *miles*, seven *wrenches* and five *bolts*.

NOTATION AND NUMERATION.

9. Numbers are expressed in three ways: (1) by words; (2) by figures; (3) by letters.

10. Notation is the art of expressing numbers by figures or letters.

11. Numeration is the art of reading the numbers which have been expressed by figures or letters.

12. The **Arabic notation** is the method of expressing numbers by figures. This method employs ten different **figures** to represent numbers, viz.:

Figures	0	1	2	3	4	5	6	7	8	9
Names	<i>naught, one cipher or zero;</i>	<i>two</i>	<i>three</i>	<i>four</i>	<i>five</i>	<i>six</i>	<i>seven</i>	<i>eight</i>	<i>nine</i>	

The first character (0) is called **naught**, **cipher**, or **zero**, and, when standing alone, has no value.

The other nine figures are called **digits**, and each one has a value of its own.

Any whole number is called an **integer**.

13. As there are only ten *figures* used in expressing numbers, each *figure* must express a different *value* at different times.

14. The value of a figure depends upon its *position* in relation to others.

15. Figures have **simple** values and **local** or **relative** values.

16. The **simple** value of a figure is the value it expresses when standing alone.

17. The **local** or **relative** value is the *increased value* it expresses by having other figures placed on its right.

For instance, if we see the figure 6 standing alone, thus. 6
we consider it as *six units*, or simply **six**.

Place another 6 to the *left* of it; thus. 66

The original figure is still *six units*, but the second one is *ten times* 6, or 6 **tens**.

If a third 6 be now placed still one place further to the *left*, it is increased in value *ten times* more, thus making it 6 **hundreds** 666

A fourth 6 would be 6 **thousands** 6666

A fifth 6 would be 6 **tens of thousands**, or **sixty thousand** 66666

A sixth 6 would be 6 **hundreds of thousands** . 666666

A seventh 6 would be 6 **millions** 6666666

The entire line of seven figures is read *six millions, six hundred sixty-six thousands, six hundred sixty-six*.

18. The **increased value** of each of these figures is its *local* or *relative* value. Each figure is *ten times* greater in value than the one immediately on its *right*.

19. The **cipher** (0) has no value itself, but it is useful in determining the place of other figures. To represent the number *four hundred five*, two digits only are necessary, one to represent *four hundred*, and the other to represent *five units*; but if these two digits are placed together, as 45, the 4 (being in the second place) will mean 4 *tens*. To mean 4 *hundreds*, the 4 should have two figures on its right, and a *cipher* is therefore inserted in the place usually given to *tens*, to show that the number is composed of *hundreds* and *units* only, and that there are no *tens*. *Four hundred five* is therefore expressed as 405. If the number were *four thousand and five*, two ciphers would be inserted; thus, 4005. If it were *four hundred fifty*, it would have the *cipher* at the right-hand side to show that there were no *units*, and only *hundreds and tens*; thus, 450. *Four thousand and fifty* would be expressed 4050, the first cipher indicating that there are no hundreds and the second that there are no units.

NOTE.—When speaking of the figures of a number by referring to them as first figure, second figure, etc., always begin to count at the *left*. Thus, in the number 41,625, 4 is the first figure, 6 the third figure, 5 the fifth or last figure, etc.

20. In *reading* figures, it is usual to point off the number into groups of three figures each, beginning with the right-hand or **units** column, a comma (,) being used to point off these groups.

<i>Billions.</i>			<i>Millions.</i>			<i>Thousands.</i>			<i>Units.</i>		
Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds of Units.	Tens of Units.	Units.
4	3	2	1	9	8	7	6	5	4	3	2

In *pointing off* these figures, begin at the right-hand figure and count—*units, tens, hundreds*; the next group of three figures is *thousands*, therefore, we insert a comma (,) before beginning with them. Beginning at the figure 5, we say *thousands, tens of thousands, hundreds of thousands*, and insert another comma; we next read *millions, tens of millions, hundreds of millions*, and insert another comma; we then read *billions, tens of billions, hundreds of billions*.

The entire line of figures would be read: *Four hundred thirty-two billions, one hundred ninety-eight millions, seven hundred sixty-five thousands, four hundred thirty-two*. When we thus *read* a line of figures it is called **numeration**, and if the **numeration** be changed back to *figures*, it is called **notation**.

For instance, the writing of the figures,

72,584,623,

would be the **notation**, and the **numeration** would be *seventy-two millions, five hundred eighty-four thousands, six hundred twenty-three*.

21. NOTE.—It is customary to leave the *s* off the words millions, thousands, etc., in cases like the above, both in speaking and writing; hence, the above would usually be expressed, *seventy-two million, five hundred eighty-four thousand, six hundred twenty-three*.

22. The four fundamental processes of Arithmetic are **addition, subtraction, multiplication, and division**. They are called fundamental processes, because all operations in Arithmetic are based upon them.

ADDITION.

23. Addition is the *process of finding the sum of two or more numbers*. The sign of addition is +. It is read *plus*, and means *more*. Thus, $5 + 6$ is read *5 plus 6*, and means that 5 and 6 are to be added.

24. The sign of equality is =. It is read *equals* or *is equal to*. Thus, $5 + 6 = 11$ may be read *5 plus 6 equals 11*.

25. *Like numbers* can be added, but *unlike numbers* cannot. Thus, 6 dollars *can* be added to 7 dollars, and the *sum* will be 13 dollars, but 6 dollars *cannot* be added to 7 feet.

26. The following table gives the sum of any two numbers from 1 to 12:

TABLE 1.

1 and 1 are 2	2 and 1 are 3	3 and 1 are 4	4 and 1 are 5
1 and 2 are 3	2 and 2 are 4	3 and 2 are 5	4 and 2 are 6
1 and 3 are 4	2 and 3 are 5	3 and 3 are 6	4 and 3 are 7
1 and 4 are 5	2 and 4 are 6	3 and 4 are 7	4 and 4 are 8
1 and 5 are 6	2 and 5 are 7	3 and 5 are 8	4 and 5 are 9
1 and 6 are 7	2 and 6 are 8	3 and 6 are 9	4 and 6 are 10
1 and 7 are 8	2 and 7 are 9	3 and 7 are 10	4 and 7 are 11
1 and 8 are 9	2 and 8 are 10	3 and 8 are 11	4 and 8 are 12
1 and 9 are 10	2 and 9 are 11	3 and 9 are 12	4 and 9 are 13
1 and 10 are 11	2 and 10 are 12	3 and 10 are 13	4 and 10 are 14
1 and 11 are 12	2 and 11 are 13	3 and 11 are 14	4 and 11 are 15
1 and 12 are 13	2 and 12 are 14	3 and 12 are 15	4 and 12 are 16
5 and 1 are 6	6 and 1 are 7	7 and 1 are 8	8 and 1 are 9
5 and 2 are 7	6 and 2 are 8	7 and 2 are 9	8 and 2 are 10
5 and 3 are 8	6 and 3 are 9	7 and 3 are 10	8 and 3 are 11
5 and 4 are 9	6 and 4 are 10	7 and 4 are 11	8 and 4 are 12
5 and 5 are 10	6 and 5 are 11	7 and 5 are 12	8 and 5 are 13
5 and 6 are 11	6 and 6 are 12	7 and 6 are 13	8 and 6 are 14
5 and 7 are 12	6 and 7 are 13	7 and 7 are 14	8 and 7 are 15
5 and 8 are 13	6 and 8 are 14	7 and 8 are 15	8 and 8 are 16
5 and 9 are 14	6 and 9 are 15	7 and 9 are 16	8 and 9 are 17
5 and 10 are 15	6 and 10 are 16	7 and 10 are 17	8 and 10 are 18
5 and 11 are 16	6 and 11 are 17	7 and 11 are 18	8 and 11 are 19
5 and 12 are 17	6 and 12 are 18	7 and 12 are 19	8 and 12 are 20
9 and 1 are 10	10 and 1 are 11	11 and 1 are 12	12 and 1 are 13
9 and 2 are 11	10 and 2 are 12	11 and 2 are 13	12 and 2 are 14
9 and 3 are 12	10 and 3 are 13	11 and 3 are 14	12 and 3 are 15
9 and 4 are 13	10 and 4 are 14	11 and 4 are 15	12 and 4 are 16
9 and 5 are 14	10 and 5 are 15	11 and 5 are 16	12 and 5 are 17
9 and 6 are 15	10 and 6 are 16	11 and 6 are 17	12 and 6 are 18
9 and 7 are 16	10 and 7 are 17	11 and 7 are 18	12 and 7 are 19
9 and 8 are 17	10 and 8 are 18	11 and 8 are 19	12 and 8 are 20
9 and 9 are 18	10 and 9 are 19	11 and 9 are 20	12 and 9 are 21
9 and 10 are 19	10 and 10 are 20	11 and 10 are 21	12 and 10 are 22
9 and 11 are 20	10 and 11 are 21	11 and 11 are 22	12 and 11 are 23
9 and 12 are 21	10 and 12 are 22	11 and 12 are 23	12 and 12 are 24

This table should be carefully committed to memory. Since 0 has no value, the sum of any number and 0 is the number itself; thus, 17 and 0 are 17.

27. For *addition*, place the numbers to be added directly under each other, taking care to place *units* under *units*, *tens* under *tens*, *hundreds* under *hundreds*, and so on.

When the numbers are thus written, the *right-hand figure* of *one number* is placed *directly under the right-hand figure*

of the *number above it*, thus bringing the unit figures of all the numbers to be added in the same vertical line. Proceed as in the following examples:

28. EXAMPLE.—What is the sum of 181, 222, 21, 2, and 418?

SOLUTION.—

$$\begin{array}{r}
 181 \\
 222 \\
 21 \\
 2 \\
 418 \\
 \hline
 \text{sum } 789 \text{ Ans.}
 \end{array}$$

EXPLANATION.—After placing the numbers in proper order, begin at the bottom of the right-hand or *units* column, and add, mentally repeating the different sums. Thus, three and two are five and one are six and two are eight and one are nine, the sum of the numbers in *units* column. Place the 9 directly beneath as the first or *units* figure in the sum.

The sum of the numbers in the next or *tens* column equals 8 *tens*, which is the second or *tens* figure in the sum.

The sum of the numbers in the next or *hundreds* column equals 7 *hundreds*, which is the third or *hundreds* figure in the sum.

The sum or answer is 789.

29. EXAMPLE.—What is the sum of 425, 36, 9,215, 4, and 907?

SOLUTION.—

$$\begin{array}{r}
 425 \\
 36 \\
 9215 \\
 4 \\
 907 \\
 \hline
 27 \\
 60 \\
 1500 \\
 9000 \\
 \hline
 \text{sum } 10587 \text{ Ans.}
 \end{array}$$

EXPLANATION.—The sum of the numbers in the first or units column is seven and four are eleven and five are sixteen and six are twenty-two and five are twenty-seven, or 27 units; i. e., two tens and seven units. Write 27 as shown.

The sum of the numbers in the second or tens column is six tens, or 60. Write 60 underneath 27 as shown. The sum of the numbers in the third or hundreds column is 15 hundreds, or 1,500. Write 1,500 under the two preceding results as shown. There is only one number in the fourth or thousands column, nine, which represents 9,000. Write 9,000 under the three preceding results. Adding these four results, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

NOTE.—It frequently happens, when adding a long column of figures, that the sum of two numbers, one of which does not occur in the addition table, is required. Thus, in the first column above, the sum of 16 and 6 was required. We know from the table that $6 + 6 = 12$; hence, the first figure of the sum is 2. Now, the sum of any number less than 20 and of any number less than 10 must be less than thirty, since $20 + 10 = 30$; therefore, the sum is 22. Consequently, in cases of this kind, add the first figure of the larger number to the smaller number and, if the result is greater than 9, increase the second figure of the larger number by 1. Thus, $44 + 7 = ?$ $4 + 7 = 11$; hence, $44 + 7 = 51$.

30. The addition may also be performed as follows:

$$\begin{array}{r}
 425 \\
 36 \\
 9215 \\
 4 \\
 907 \\
 \hline
 \text{sum } 10587 \text{ Ans.}
 \end{array}$$

EXPLANATION.—The sum of the numbers in *units* column = 27 *units*, or 2 *tens* and 7 *units*. Write the 7 *units* as the first or right-hand figure in the sum. Reserve the two *tens* and add them to the figures in *tens* column. The sum of the figures in the *tens* column, plus the 2 *tens* reserved and carried from the *units* column = 8, which is written down as the second figure in the sum. There is nothing to carry to the next column, because 8 is less than 10. The sum of the numbers in the next column is 15 *hundreds*, or 1 *thousand* and 5 *hundreds*. Write down the 5 as the third or *hundreds* figure in the sum and carry the 1 to the next column. $1 + 9 = 10$, which is written down at the left of the other figures.

The second method saves space and figures, but the first is to be preferred when adding a long column.

31. EXAMPLE.—Add the numbers in the column below.

SOLUTION.—

$$\begin{array}{r}
 890 \\
 82 \\
 90 \\
 393 \\
 281 \\
 80 \\
 770 \\
 83 \\
 492 \\
 80 \\
 383 \\
 84 \\
 191 \\
 \hline
 \text{sum } 3899 \quad \text{Ans.}
 \end{array}$$

EXPLANATION.—The sum of the digits in the first column equals 19 *units*, or 1 *ten* and 9 *units*. Write down the 9 and carry 1 to the next column. The sum of the digits in the second column + 1 = 109 *tens*, or 10 *hundreds* and 9 *tens*. Write down the 9 and carry the 10 to the next column. The sum of the digits in this column plus the 10 reserved = 38.

The entire sum is 3,899.

32. Rule 1.—(a) *Begin at the right, add each column separately, and write the sum, if it be only one figure, under the column added.*

(b) *If the sum of any column consists of two or more figures, put the right-hand figure of the sum under that column, and add the remaining figure or figures to the next column.*

33. Proof.—*To prove addition, add each column from top to bottom. If you obtain the same result as by adding from bottom to top, the work is probably correct.*

EXAMPLES FOR PRACTICE.

34. Find the sum of

- (a) $104 + 203 + 618 + 214.$
 (b) $1,875 + 3,143 + 5,826 + 10,832.$
 (c) $4,865 + 2,145 + 8,173 + 40,084.$
 (d) $14,204 + 8,173 + 1,065 + 10,042.$

Ans. $\left\{ \begin{array}{l} (a) \ 1,184. \\ (b) \ 21,676. \\ (c) \ 55,267. \\ (d) \ 33,484. \end{array} \right.$

$$\begin{array}{ll}
 (e) & 10,882 + 4,145 + 3,133 + 5,872. \\
 (f) & 214 + 1,281 + 141 + 5,000. \\
 (g) & 123 + 104 + 425 + 126 + 327. \\
 (h) & 6,354 + 2,145 + 2,042 + 1,111 + 3,833.
 \end{array}
 \quad \text{Ans. } \left\{ \begin{array}{l} (e) \ 23,982. \\ (f) \ 6,580. \\ (g) \ 1,105. \\ (h) \ 14,985. \end{array} \right.$$

1. A week's record of coal burned in an engine room is as follows : Monday, 1,800 pounds ; Tuesday, 1,655 pounds ; Wednesday, 1,725 pounds ; Thursday, 1,690 pounds ; Friday, 1,648 pounds ; Saturday, 1,020 pounds. How much coal was burned during the week ?

Ans. 9,538 pounds.

2. A steam pump pumps out of a cistern in one hour 4,200 gallons ; in the next hour, 5,420 gallons, and in 45 minutes more, an additional 3,600 gallons, when the cistern becomes empty. How many gallons were in the cistern at first ?

Ans. 13,220 gallons.

3. What is the total cost of a steam plant, the several items of expense being as follows : Steam engine, \$900 ; boiler, \$775 ; fittings and connections, \$225 ; erecting the plant, \$125 ; engine house, \$650 ?

Ans. \$2,675.

SUBTRACTION.

35. In Arithmetic, **subtraction** is the process of finding how much greater one number is than another.

The greater of the two numbers is called the **minuend**.

The smaller of the two numbers is called the **subtrahend**.

The number left after subtracting the *subtrahend* from the *minuend* is called the **difference** or **remainder**.

36. The sign of subtraction is —. It is read **minus**, and means *less*. Thus, $12 - 7$ is read 12 *minus* 7, and means that 7 is to be taken from 12.

37. EXAMPLE.—From 7,568 take 3,425.

SOLUTION.—

$$\begin{array}{r}
 \text{minuend } 7568 \\
 \text{subtrahend } 3425 \\
 \hline
 \text{remainder } 4143 \text{ Ans.}
 \end{array}$$

EXPLANATION.—Begin at the right-hand or *units* column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

38. When there are more figures in the *minuend* than in the *subtrahend*, and when some figures in the *minuend* are less than the figures directly under them in the *subtrahend*, proceed as in the following example:

EXAMPLE.—From 8,453 take 844.

$$\begin{array}{r} \text{SOLUTION.} \quad \text{minuend } 8453 \\ \text{subtrahend } 844 \\ \hline \text{remainder } 7609 \text{ Ans.} \end{array}$$

EXPLANATION.—Begin at the right-hand or *units* column to subtract. We can not take 4 from 3, and must, therefore, borrow 1 from 5 in *tens* column and annex it to the 3 in *units* column. The 1 *ten* = 10 *units*, which added to the 3 in the *units* column = 13 *units*. 4 from 13 = 9, the first or *units* figure in the remainder.

Since we borrowed 1 from the 5, only 4 remains; 4 from 4 = 0, the second or *tens* figure. We can not take 8 from 4, so borrow 1 *thousand* or 10 *hundreds* from 8; 10 *hundreds* + 4 *hundreds* = 14 *hundreds*.

8 from 14 = 6, the third or *hundreds* figure in the remainder.

Since we borrowed 1 from 8 only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder or answer.

The operation of borrowing is placing 1 before the figure following the one from which it is borrowed. In the above example the one borrowed from 5 is placed before 3, making it 13, from which we subtract 4. The 1 borrowed from 8 is placed before 4, making 14, from which 8 is taken.

39. **EXAMPLE.**—Find the difference between 10,000 and 8,763.

$$\begin{array}{r} \text{SOLUTION.} \quad \text{minuend } 10000 \\ \text{subtrahend } 8763 \\ \hline \text{remainder } 1237 \text{ Ans.} \end{array}$$

EXPLANATION.—In the above example we borrow 1 from the second column and place it before 0, making 10; 3 from 10 = 7. In the same way we borrow 1 and place it before

the next cipher, making 10; but as we have borrowed 1 from this column and taken it to the *units* column, only 9 remains, from which to subtract 6, 6 from $9 = 3$. For the same reason we subtract 7 from 9 and 8 from 9 for the next two figures, and obtain a total remainder of 1,237.

40. Rule 2.—Place the subtrahend or smaller number under the minuend or larger number, in the same manner as for addition, and proceed as in Arts. 37, 38, and 39.

41. Proof.—To prove an example in subtraction, add the subtrahend and remainder. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.

Proof of the above example:

$$\begin{array}{r} \text{subtrahend} \quad 8763 \\ \text{remainder} \quad 1237 \\ \hline \text{minuend} \quad 10000 \end{array}$$

EXAMPLES FOR PRACTICE.

42. From

(a) 94,278 take 62,574.

(b) 53,714 take 25,824.

(c) 71,832 take 58,109.

(d) 20,804 take 10,408.

(e) 810,465 take 102,141.

(f) $(81,043 + 1,041)$ take 14,881.

(g) $(20,482 + 18,216)$ take 21,214.

(h) $(2,040 + 1,218 + 542)$ take 3,791.

Ans. $\left\{ \begin{array}{l} (a) 31,704. \\ (b) 27,890. \\ (c) 13,723. \\ (d) 10,396. \\ (e) 208,324. \\ (f) 67,253. \\ (g) 17,484. \\ (h) 4. \end{array} \right.$

1. A cistern is fed by two pipes which supply 1,200 and 2,250 gallons per hour, respectively, and is being emptied by a pump which delivers 5,800 gallons per hour. Starting with 8,000 gallons in the cistern, how much water does it contain at the end of an hour? Ans. 5,650 gallons.

2. A train in running from New York to Buffalo travels 38 miles the first hour, 42 the second, 39 the third, 56 the fourth, 52 the fifth, and 48 the sixth hour. How many miles remain to be traveled at the end of the sixth hour, the distance between the two places being 410 miles? Ans. 135 miles.

3. On Monday morning a bank had on hand \$2,802. During the day \$1,831 were deposited and \$2,172 drawn out; on Tuesday, \$3,126 were deposited, and \$1,954 drawn out. How many dollars were on hand Wednesday morning? Ans. \$3,693.

MULTIPLICATION.

43. To **multiply** a number is to *add* it to itself a certain number of times.

44. **Multiplication** is the process of multiplying one number by another.

The *number* thus added to itself, or the number to be multiplied, is called the **multiplicand**.

The *number* which shows how many times the *multiplicand* is to be taken, or the *number* by which we *multiply*, is called the **multiplier**.

The result obtained by multiplying is called the **product**.

45. The sign of multiplication is $\times$. It is read *times* or *multiplied by*. Thus, 9×6 is read 9 *times* 6, or 9 *multiplied by* 6.

46. It matters not in what order the numbers to be multiplied together are placed. Thus, 6×9 is the **same as** 9×6 .

47. In the following table, the product of any two numbers (neither of which exceeds twelve) may be found:

TABLE 2.

1 times 1 are 1	2 times 1 are 2	3 times 1 are 3
1 times 2 are 2	2 times 2 are 4	3 times 2 are 6
1 times 3 are 3	2 times 3 are 6	3 times 3 are 9
1 times 4 are 4	2 times 4 are 8	3 times 4 are 12
1 times 5 are 5	2 times 5 are 10	3 times 5 are 15
1 times 6 are 6	2 times 6 are 12	3 times 6 are 18
1 times 7 are 7	2 times 7 are 14	3 times 7 are 21
1 times 8 are 8	2 times 8 are 16	3 times 8 are 24
1 times 9 are 9	2 times 9 are 18	3 times 9 are 27
1 times 10 are 10	2 times 10 are 20	3 times 10 are 30
1 times 11 are 11	2 times 11 are 22	3 times 11 are 33
1 times 12 are 12	2 times 12 are 24	3 times 12 are 36
4 times 1 are 4	5 times 1 are 5	6 times 1 are 6
4 times 2 are 8	5 times 2 are 10	6 times 2 are 12
4 times 3 are 12	5 times 3 are 15	6 times 3 are 18
4 times 4 are 16	5 times 4 are 20	6 times 4 are 24
4 times 5 are 20	5 times 5 are 25	6 times 5 are 30
4 times 6 are 24	5 times 6 are 30	6 times 6 are 36
4 times 7 are 28	5 times 7 are 35	6 times 7 are 42
4 times 8 are 32	5 times 8 are 40	6 times 8 are 48
4 times 9 are 36	5 times 9 are 45	6 times 9 are 54
4 times 10 are 40	5 times 10 are 50	6 times 10 are 60
4 times 11 are 44	5 times 11 are 55	6 times 11 are 66
4 times 12 are 48	5 times 12 are 60	6 times 12 are 72
7 times 1 are 7	8 times 1 are 8	9 times 1 are 9
7 times 2 are 14	8 times 2 are 16	9 times 2 are 18
7 times 3 are 21	8 times 3 are 24	9 times 3 are 27
7 times 4 are 28	8 times 4 are 32	9 times 4 are 36
7 times 5 are 35	8 times 5 are 40	9 times 5 are 45
7 times 6 are 42	8 times 6 are 48	9 times 6 are 54
7 times 7 are 49	8 times 7 are 56	9 times 7 are 63
7 times 8 are 56	8 times 8 are 64	9 times 8 are 72
7 times 9 are 63	8 times 9 are 72	9 times 9 are 81
7 times 10 are 70	8 times 10 are 80	9 times 10 are 90
7 times 11 are 77	8 times 11 are 88	9 times 11 are 99
7 times 12 are 84	8 times 12 are 96	9 times 12 are 108
10 times 1 are 10	11 times 1 are 11	12 times 1 are 12
10 times 2 are 20	11 times 2 are 22	12 times 2 are 24
10 times 3 are 30	11 times 3 are 33	12 times 3 are 36
10 times 4 are 40	11 times 4 are 44	12 times 4 are 48
10 times 5 are 50	11 times 5 are 55	12 times 5 are 60
10 times 6 are 60	11 times 6 are 66	12 times 6 are 72
10 times 7 are 70	11 times 7 are 77	12 times 7 are 84
10 times 8 are 80	11 times 8 are 88	12 times 8 are 96
10 times 9 are 90	11 times 9 are 99	12 times 9 are 108
10 times 10 are 100	11 times 10 are 110	12 times 10 are 120
10 times 11 are 110	11 times 11 are 121	12 times 11 are 132
10 times 12 are 120	11 times 12 are 132	12 times 12 are 144

This table should be carefully committed to memory.

Since 0 has no value, the product of 0 and any number is 0.

48. To multiply a number by one figure only :

EXAMPLE.—Multiply 425 by 5.

$$\begin{array}{r}
 \text{multiplicand} \quad 425 \\
 \text{multiplier} \quad \quad 5 \\
 \hline
 \text{product} \quad 2125 \quad \text{Ans.}
 \end{array}$$

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *right-hand figure* of the *multiplicand*. On looking in the multiplication table, we see that 5×5 are 25. *Multiplying* the *first figure* at the *right* of the *multiplicand*, or 5, by the *multiplier* 5, it is seen that 5 times 5 units are 25 units, or 2 tens and 5 units. Write the 5 *units* in *units place* in the *product*, and *reserve* the 2 tens to *add* to the *product* of *tens*. Looking in the multiplication table again, we see that 5×2 are 10. *Multiplying* the *second figure* of the *multiplicand* by the *multiplier* 5, we see that 5 times 2 tens are 10 tens, *plus* the 2 tens *reserved*, are 12 tens, or 1 hundred plus 2 tens. Write the 2 *tens* in *tens place*, and *reserve* the 1 hundred to *add* to the *product* of *hundreds*. Again, we see by the multiplication table that 5×4 are 20. *Multiplying* the *third* or *last figure* of the *multiplicand* by the *multiplier* 5, we see that 5 times 4 hundreds are 20 hundreds, *plus* the 1 hundred *reserved*, are 21 hundreds, or 2 thousands *plus* 1 hundred, which we write in *thousands* and *hundreds places*, respectively.

Hence, the *product* is 2,125.This *result* is the same as adding 425 five times. Thus,

$$\begin{array}{r}
 425 \\
 425 \\
 425 \\
 425 \\
 425 \\
 \hline
 \text{sum} \quad 2125 \quad \text{Ans.}
 \end{array}$$

EXAMPLES FOR PRACTICE.**49.** Find the product of

- (a) $61,483 \times 6$.
 (b) $12,375 \times 5$.
 (c) $10,426 \times 7$.
 (d) $10,835 \times 3$.

$$\text{Ans.} \left\{ \begin{array}{ll} (a) & 368,898. \\ (b) & 61,875. \\ (c) & 72,982. \\ (d) & 32,505. \end{array} \right.$$

$$\begin{array}{ll}
 (e) & 98,376 \times 4 \\
 (f) & 10,878 \times 8 \\
 (g) & 71,543 \times 9 \\
 (h) & 218,734 \times 2
 \end{array}
 \quad
 \text{Ans. } \left\{ \begin{array}{l}
 (e) \quad 393,504 \\
 (f) \quad 86,984 \\
 (g) \quad 643,887 \\
 (h) \quad 437,468
 \end{array} \right.$$

1. A stationary engine makes 5,520 revolutions per hour. Running 9 hours a day, 5 days in the week, and 5 hours on Saturday, how many revolutions would it make in 4 weeks? Ans. 1,104,000 revolutions.

2. An engineer earns \$650 a year, and his average expenses are \$548. How much could he save in 8 years at that rate? Ans. \$816.

3. The connection between an engine and boiler is made up of 5 lengths of pipe, three of which are 12 feet long, one 2 feet 6 inches long, and one 8 feet 6 inches long. If the pipe weighs 9 pounds per foot, what is the total weight of the pipe used? Ans. 423 pounds.

50. To multiply a number by two or more figures:

EXAMPLE.—Multiply 475 by 234.

$$\begin{array}{r}
 \text{SOLUTION.—} \quad \begin{array}{l} \text{multiplicand} \\ \text{multiplier} \end{array} \quad \begin{array}{r} 475 \\ 234 \\ \hline 1900 \\ 1425 \\ 950 \\ \hline \text{product } 111150 \end{array} \quad \text{Ans.}
 \end{array}$$

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *multiplicand*, placing units under units, tens under tens, etc.

We *can not* multiply by 234 at one operation; we must, therefore, *multiply* by the *parts* and then *add* the **partial products**.

The parts by which we are to multiply are 4 units, 3 tens, and 2 hundreds. 4 times 475 = 1,900, the *first partial product*; 3 times 475 = 1,425, the *second partial product*, the *right-hand figure* of which is *written directly under the figure multiplied by*, or 3; 2 times 475 = 950, the *third partial product*, the *right-hand figure* of which is *written directly under the figure multiplied by*, or 2.

The sum of these *three partial products* is 111,150, which is the *entire product*.

51. Rule 3.—(a) Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.

(b) Begin at the right and multiply each figure of the multiplicand by each successive figure of the multiplier, placing the right-hand figure of each partial product directly under the figure used as a multiplier.

(c) The sum of the partial products will equal the required product.

52. Proof.—Review the work carefully, or multiply the multiplier by the multiplicand; if the results agree, the work is correct.

53. When there is a *cipher* in the multiplier, multiply the entire multiplicand by it; since the result will be zero, place a cipher under the cipher in the multiplier. Thus,

(a) $\begin{array}{r} 0 \\ \times 0 \\ \hline 0 \end{array}$	(b) $\begin{array}{r} 2 \\ \times 0 \\ \hline 0 \end{array}$	(c) $\begin{array}{r} 15 \\ \times 0 \\ \hline 0 \end{array}$	(d) $\begin{array}{r} 708 \\ \times 0 \\ \hline 0 \end{array}$
0 Ans.	0 Ans.	0 Ans.	0 Ans.

(e) $\begin{array}{r} 8114 \\ 203 \\ \hline 9342 \\ 62280 \\ \hline 632142 \end{array}$	(f) $\begin{array}{r} 4008 \\ 305 \\ \hline 20040 \\ 120240 \\ \hline 1222440 \end{array}$	(g) $\begin{array}{r} 31264 \\ 1002 \\ \hline 62528 \\ 3126400 \\ \hline 31326528 \end{array}$
632142 Ans.	1222440 Ans.	31326528 Ans.

In examples (e), (f), and (g), we multiply by 0 as directed above; then multiply by the next figure of the multiplier and place the first figure of the product alongside the 0, as shown.

EXAMPLES FOR PRACTICE.

54. Find the product of

(a) $3,842 \times 26.$ (b) $3,716 \times 45.$ (c) $1,817 \times 124.$ (d) $675 \times 38.$	Ans. $\left\{ \begin{array}{l} (a) \ 99,892. \\ (b) \ 167,220. \\ (c) \ 225,308. \\ (d) \ 25,650. \end{array} \right.$
---	--

(e) $1,875 \times 33.$	Ans.	(e) 61,875.
(f) $4,836 \times 47.$		(f) 227,292.
(g) $5,682 \times 543.$		(g) 3,085,326.
(h) $3,257 \times 246.$		(h) 801,222.
(i) $2,875 \times 302.$		(i) 868,250.
(j) $17,819 \times 1,004.$		(j) 17,890,276.
(k) $38,674 \times 205.$		(k) 7,928,170.
(l) $18,304 \times 100.$		(l) 1,830,400.
(m) $7,834 \times 10.$		(m) 78,340.
(n) $87,543 \times 1,000.$		(n) 87,543,000.
(o) $48,768 \times 100.$		(o) 4,876,800.

1. If the area of a steam-engine piston is 113 square inches, what is the total pressure upon it when the steam pressure is 85 pounds per square inch?
Ans. 9,605 pounds.

2. A steam engine, which indicated 164 horsepower, was found to consume 4 pounds of coal per horsepower per hour. Being replaced by a new engine, which was of the same horsepower as the other, another test was made, which showed a consumption of 3 pounds per horsepower per hour. What was the saving of coal for a year of 309 days, if the engine averaged to run 14 hours a day?
Ans. 709,464 pounds.

3. Two steamers are 7,846 miles apart, and are sailing towards each other, one at the rate of 18 miles an hour, and the other at the rate of 15 miles an hour. How far apart will they be at the end of 205 hours?
Ans. 1,081 miles.

DIVISION.

55. **Division** is the process of finding how many times one number is contained in another of the same kind.

The number to be *divided* is called the **dividend**.

The number by which we *divide* is called the **divisor**.

The number which *shows* how many times the *divisor* is contained in the *dividend* is called the **quotient**.

56. The sign of division is $\div$. It is read *divided by*. $54 \div 9$ is read 54 *divided by* 9. Another way to write 54 *divided by* 9 is $\frac{54}{9}$. Thus, $54 \div 9 = 6$, or $\frac{54}{9} = 6$.

In both of these cases 54 is the *dividend* and 9 is the *divisor*.

Division is the *reverse* of **multiplication**.

57. To divide when the divisor consists of but one figure, proceed as in the following example:

EXAMPLE.—What is the quotient of $875 \div 7$?

SOLUTION.—

$$\begin{array}{r}
 \text{divisor} \quad \text{dividend} \quad \text{quotient} \\
 7 \overline{) 875} \quad 125 \quad \text{Ans.} \\
 \underline{7} \\
 17 \\
 \underline{14} \\
 35 \\
 \underline{35} \\
 0
 \end{array}$$

remainder 0

EXPLANATION.—7 is contained in 8 *hundreds* 1 *hundred* times. Place the one as the first, or left-hand, figure of the quotient. Multiply the divisor 7 by the 1 *hundred* of the quotient, and place the product 7 *hundreds* under the 8 *hundreds* in the dividend, and subtract. Beside the remainder 1, bring down the 7 *tens*, making 17 *tens*; 17 divided by 7 = 2 times. Write the two as the second figure of the quotient. Multiply the divisor 7 by the 2, and subtract the product from 17. Beside the remainder 3, bring down the 5 *units* of the dividend, making 35 *units*. 7 is contained in 35, 5 times, which is placed in the quotient. Multiplying the divisor by the last figure of the quotient, 5 times 7 = 35, which subtracted from 35, under which it is placed, leaves 0. Therefore, the quotient is 125. This method is called **long division**.

58. In short division, only the divisor, dividend, and quotient are written, the operations being performed mentally.

$$\begin{array}{r}
 \text{dividend} \\
 \text{divisor} \quad 7 \overline{) 875} \\
 \text{quotient} \quad 125 \quad \text{Ans.}
 \end{array}$$

The mental operation is as follows: 7 is contained in 8, once and one remainder; 1 placed before 7 makes 17; 7 is contained in 17, 2 times and 3 over; the 3 placed before 5 makes 35; 7 is contained in 35, 5 times. These partial quotients placed in order as they are found, make the entire quotient 125.

The small figures are placed in the example given to better illustrate the explanation; they are never written when actually performing division in this way.

59. If the *divisor* consists of 2 or more figures, proceed as in the following example:

EXAMPLE.—Divide 2,702,826 by 63.

<i>divisor</i>	<i>dividend</i>	<i>quotient</i>	
63	2702826	(42902	Ans.
	252		
	182		
	126		
	568		
	567		
	126		
	126		
	0		

EXPLANATION.—As 63 is not contained in the first two figures, 27, we must use the first three figures, 270. Now, by trial, we must find how many times 63 is contained in 270; 6 is contained in the first two figures of 270, 4 times. Place the 4 as the first or left-hand figure in the quotient. Multiply the divisor 63 by 4, and subtract the product 252 from 270. The remainder is 18, beside which we write the next figure of the dividend, 2, making 182. Now, 6 is contained in the first two figures of 182, 3 times, but on multiplying 63 by 3, we see that the product 189 is too great, so we try 2 as the second figure of the quotient. Multiplying the divisor 63 by 2, and subtracting the product 126 from 182, the remainder is 56, beside which we bring down the next figure of the dividend, making 568; 6 is contained in 56 about 9 times. Multiply the divisor 63 by 9 and subtract the product 567 from 568. The remainder is 1, and bringing down the next figure of the dividend, 2, gives 12. As 12 is smaller than 63, we write 0 in the quotient and bring down the next figure, 6, making 126. 63 is contained in 126, 2 times, without a remainder. Therefore, 42,902 is the quotient.

60. Rule 4.—(a) Write the divisor at the left of the dividend, with a line between them.

(b) Find how many times the divisor is contained in the lowest number of the left-hand figures of the dividend that will contain it, and write the result at the right of the dividend, with a line between, for the first figure of the quotient.

(c) Multiply the divisor by this quotient; write the product under the partial dividend used, and subtract, annexing to the remainder the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.

(d) If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend and proceed as before.

(e) If there be a remainder at last, write it after the quotient, with the divisor underneath.

61. Proof.—Multiply the quotient by the divisor, and add the remainder, if there be any, to the product. The result will be the dividend.

	divisor	dividend	quotient	
Thus,	68	4235	(67 $\frac{14}{68}$)	Ans.
		378		
		455		
		441		
	remainder	14		
Proof,	quotient	67		
	divisor	68		
		201		
		402		
		4221		
	remainder	14		
	dividend	4235		

EXAMPLES FOR PRACTICE.

62. Divide the following:

(a) 126,498 by 58.	Ans. {	(a) 2,181.
(b) 3,207,594 by 767.		(b) 4,182.
(c) 11,408,202 by 234.		(c) 48,753.
(d) 2,100,315 by 581.		(d) 3,615.
(e) 969,936 by 4,008.		(e) 242.
(f) 7,481,888 by 1,021.		(f) 7,328.
(g) 1,525,915 by 5,008.		(g) 305.
(h) 1,646,301 by 381.		(h) 4,321.

1. In a mile there are 5,280 feet. How many rails would it take to lay a double row one mile long, each rail being 30 feet long?

Ans. 352 rails.

2. How many rivets will be required for the longitudinal seams of a cylindrical boiler 20 feet long, the joint being double riveted, and the rivets being spaced 4 inches apart?

Ans. 120 rivets.

NOTE.—First find the length of the boiler in inches.

3. It requires 7,020,000 bricks to build a large foundry. How many teams will it require to draw the bricks in 60 days, if each team draws 6 loads per day and 1,500 bricks at a load?

Ans. 13 teams.

NOTE.—Find how many loads 7,020,000 bricks make; then, how many days it will take one team to draw the brick.

CANCELLATION.

63. Cancellation is the process of shortening operations in division by casting out equal factors from both dividend and divisor.

64. The **factors** of a number are *those numbers* which, when *multiplied* together, will *equal that number*. Thus, 5 and 3 are factors of 15, since $5 \times 3 = 15$. Likewise, 8 and 7 are the factors of 56, since $8 \times 7 = 56$.

65. A **prime number** is one which can not be divided by any number except itself and 1. Thus, 2, 3, 11, 29, etc., are prime numbers.

66. A **prime factor** is any factor that is a prime number.

Any number that is not a prime is called a **composite** number, and may be produced by multiplying together its prime factors. Thus, 60 is a composite number, and is equal to the product of its prime factors, $2 \times 2 \times 3 \times 5$.

Numbers are said to be **prime to each other** when no two of them can be divided by any number except 1; the numbers themselves *may* be either prime or composite. Thus, the numbers 3, 5, and 11 are prime to each other, so also are 22, 25, and 21 — all composite numbers.

67. Canceling *equal factors* from *both dividend and divisor* does *not* change the *quotient*.

The *canceling* of a *factor* in *both dividend and divisor* is the *same* as *dividing them both* by the *same number*, which, by the principle of division, does not *change the quotient*.

Write the *numbers* which make the *dividend* above the *line*, and those which make the *divisor* below it.

68. EXAMPLE.—Divide $4 \times 45 \times 60$ by 9×24 .

SOLUTION.—Placing the dividend over the divisor, and canceling

$$\begin{array}{r} 5 \quad 10 \\ 4 \times \cancel{45} \times \cancel{60} \\ 9 \times \cancel{24} \end{array} = \frac{50}{1} = 50. \quad \text{Ans.}$$

6
1

EXPLANATION.—The 4 in the dividend and 24 in the divisor are both divisible by 4, since 4 divided by 4 equals 1, and 24 divided by 4 equals 6. Cross off the four and write the 1 over it; also, cross off the 24 and write the 6 under it. Thus,

$$\begin{array}{r} 1 \\ 4 \times 45 \times 60 \\ 9 \times \cancel{24} \\ 6 \end{array}$$

60 in the dividend and 6 in the divisor are divisible by 6, since 60 divided by 6 equals 10, and 6 divided by 6 equals 1. Cross off the 60 and write 10 over it; also, cross off the 6 and write 1 under it. Thus,

$$\begin{array}{r} 1 \quad 10 \\ 4 \times 45 \times \cancel{60} \\ 9 \times \cancel{24} \\ 6 \\ 1 \end{array}$$

Again, 45 in the dividend and 9 in the divisor are divisible by 9, since 45 divided by 9 equals 5, and 9 divided by 9 equals 1. Cross off the 45 and write the 5 over it; also, cross off the 9 and write the 1 under it. Thus,

$$\begin{array}{r} 1 \quad 5 \quad 10 \\ 4 \times \cancel{45} \times \cancel{60} \\ 9 \times \cancel{24} \\ 1 \quad 6 \\ 1 \end{array}$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together, and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $5 \times 1 \times 10 = 50$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 = 1$.

$$\text{Hence, } \frac{\overset{1}{4} \times \overset{5}{45} \times \overset{10}{60}}{\underset{1}{9} \times \underset{\underset{1}{6}}{24}} = \frac{1 \times 5 \times 10}{1 \times 1} = 50. \quad \text{Ans.}$$

It is usual to omit the 1's when canceling them, instead of writing them as above.

69. Rule 5.—(a) *Cancel the common factors from both the dividend and divisor.*

(b) *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the quotient.*

EXAMPLES FOR PRACTICE.

70. Divide

- (a) $14 \times 18 \times 16 \times 40$ by $7 \times 8 \times 6 \times 5 \times 3$.
 (b) $3 \times 65 \times 50 \times 100 \times 60$ by $30 \times 60 \times 13 \times 10$.
 (c) $8 \times 4 \times 3 \times 9 \times 11$ by $11 \times 9 \times 4 \times 3 \times 8$.
 (d) $164 \times 321 \times 6 \times 7 \times 4$ by $82 \times 321 \times 7$.
 (e) $50 \times 100 \times 200 \times 72$ by $1,000 \times 144 \times 100$.
 (f) $48 \times 63 \times 55 \times 49$ by $7 \times 21 \times 11 \times 48$.
 (g) $110 \times 150 \times 84 \times 32$ by $11 \times 15 \times 100 \times 64$.
 (h) $115 \times 120 \times 400 \times 1,000$ by $23 \times 1,000 \times 60 \times 800$.

Ans. $\left\{ \begin{array}{l} (a) 32. \\ (b) 250. \\ (c) 1. \\ (d) 48. \\ (e) 5. \\ (f) 105. \\ (g) 42. \\ (h) 5. \end{array} \right.$

FRACTIONS.

71. A **fraction** is a *part of a whole number*: *One-half, one-third, two-fifths* are fractions.

72. Two numbers are required to express a fraction, one called the **numerator**, and the other the **denominator**.

73. The *numerator* is placed above the *denominator*, with a *line* between them; as, $\frac{1}{3}$. 3 is the *denominator*, and shows into how many *equal parts* the *unit* or *one* is divided. The *numerator* 2 shows how many of these *equal parts* are taken or considered. The *denominator* also indicates the *names* of the parts.

$\frac{1}{2}$ is read one-half.

$\frac{3}{4}$ is read three-fourths.

$\frac{3}{8}$ is read three-eighths.

$\frac{5}{16}$ is read five-sixteenths.

$\frac{29}{47}$ is read twenty-nine-forty-sevenths.

In the expression " $\frac{3}{4}$ of an apple," the *denominator* 4 shows that the apple is to be (or has been) cut into 4 *equal parts*, and the *numerator* 3 shows that *three of these parts*, or *fourths*, are taken or considered.

If each of the *parts*, or *fourths*, of the apple were cut in *two equal pieces*, there would then be twice as many pieces as before, or $4 \times 2 = 8$ pieces in all; one of these pieces would be called one-eighth, and would be expressed in figures as $\frac{1}{8}$. Three of these pieces would be called three-eighths, and written $\frac{3}{8}$. The words three-fourths, three-eighths, five-sixteenths, etc., are abbreviations of three one-fourths, three one-eighths, five one-sixteenths, etc. It is evident that the larger the *denominator*, the greater is the number of *parts* into which anything is divided; consequently, the *parts* themselves are smaller, and the value of the fraction is less for the same number of parts taken. In other words, $\frac{1}{9}$, for example, is smaller than $\frac{1}{8}$, because if an object be divided into 9 parts, the parts are smaller than if the same object had been divided into 8 parts; and, since $\frac{1}{9}$ is smaller than $\frac{1}{8}$, it is clear that 7 one-ninths is a smaller amount than 7 one-eighths. Hence, also, $\frac{7}{9}$ is less than $\frac{7}{8}$.

74. The *value* of a fraction is the result obtained by dividing the *numerator* by the *denominator*; as, $\frac{4}{2} = 2$, $\frac{9}{3} = 3$.

75. The line between the *numerator* and *denominator* means *divided by*, or $\div$.

$\frac{3}{4}$ is equivalent to $3 \div 4$.

$\frac{5}{8}$ is equivalent to $5 \div 8$.

76. The *numerator* and *denominator* of a fraction are called the **terms** of a fraction.

77. The *value* of a fraction whose *numerator* and *denominator* are equal, is 1.

$\frac{4}{4}$, or four-fourths, = 1.

$\frac{8}{8}$, or eight-eighths, = 1.

$\frac{64}{64}$, or sixty-four sixty-fourths, = 1.

78. A **proper fraction** is a fraction whose *numerator* is *less* than its *denominator*. Its *value* is *less* than 1; as, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{16}$.

79. An **improper fraction** is a fraction whose *numerator* *equals* or is *greater* than the *denominator*. Its *value* is *one* or *more* than *one*; as, $\frac{4}{4}$, $\frac{9}{8}$, $\frac{11}{2}$.

80. A **mixed number** is a *whole* number and a *fraction* united. $4\frac{2}{3}$ is a mixed number, and is equivalent to $4 + \frac{2}{3}$. It is read *four and two-thirds*.

REDUCTION OF FRACTIONS.

81. **Reduction of fractions** is the process of changing their form without changing their *value*.

82. A *fraction* is reduced to *higher terms* by *multiplying both terms* of the *fraction* by the *same number*. Thus, $\frac{3}{4}$ is reduced to $\frac{6}{8}$ by multiplying both terms by 2.

$$\frac{3}{4} \times 2 = \frac{6}{8}.$$

The *value* is not changed, since $\frac{3}{4} = \frac{6}{8}$. For, suppose that an object, say an apple, is divided into 8 equal parts. If these parts be arranged into 4 piles, each containing 2 parts, it is evident each pile will be composed of the same amount of the entire apple as would have been the case had the apple been originally cut into 4 equal parts. Now, if one of these piles (containing 2 parts) be removed, there will be 3 piles left each containing 2 equal parts, or 6 equal parts in all, i. e., six-eighths. But, since one pile, or one quarter, was removed, there are three-quarters left. Hence, $\frac{3}{4} = \frac{6}{8}$.

The same course of reasoning may be applied to any similar case. Therefore, multiplying both terms of a fraction by the same number does not alter its value.

$$\frac{3}{4} \times 2 = \frac{6}{8}$$

83. A fraction is reduced to *lower terms* by dividing both terms by the same number. Thus, $\frac{8}{10}$ is reduced to $\frac{4}{5}$ by dividing both terms by 2.

$$\frac{8 \div 2}{10 \div 2} = \frac{4}{5}$$

That $\frac{8}{10} = \frac{4}{5}$ is readily seen from the explanation given in Art. 82; for, multiplying both terms of the fraction $\frac{4}{5}$ by 2, $\frac{4}{5} \times \frac{2}{2} = \frac{8}{10}$, and, if $\frac{4}{5} = \frac{8}{10}$, $\frac{8}{10}$ must equal $\frac{4}{5}$. Hence, dividing both terms of a fraction by the same number does not alter its value.

84. A fraction is reduced to its *lowest terms* when its numerator and denominator can not be divided by the same number, as $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{11}$.

85. To reduce a whole number or mixed number to an improper fraction:

EXAMPLE.—How many *fourths* in 5?

SOLUTION.—Since there are 4 *fourths* in 1 ($\frac{4}{4} = 1$), in 5 there will be 5×4 fourths, or 20 fourths, $5 \times \frac{4}{4} = \frac{20}{4}$. Ans.

EXAMPLE.—Reduce $8\frac{3}{4}$ to an improper fraction.

SOLUTION.— $8 \times \frac{4}{4} = \frac{32}{4}$. $\frac{32}{4} + \frac{3}{4} = \frac{35}{4}$. Ans.

86. Rule 6.—Multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the denominator under the result.

EXAMPLES FOR PRACTICE.

87. Reduce to improper fractions:

- (a) $4\frac{1}{2}$.
- (b) $5\frac{1}{3}$.
- (c) $10\frac{3}{10}$.
- (d) $37\frac{1}{2}$.
- (e) $50\frac{1}{4}$.

$$\text{Ans. } \left\{ \begin{array}{l} (a) \frac{9}{2}. \\ (b) \frac{16}{3}. \\ (c) \frac{103}{10}. \\ (d) \frac{141}{2}. \\ (e) \frac{201}{4}. \end{array} \right.$$

88. To reduce an improper fraction to a whole or mixed number :

EXAMPLE.—Reduce $2\frac{1}{4}$ to a mixed number.

SOLUTION.—4 is contained in 21, 5 times and 1 remaining; as this is also divided by 4, its value is $\frac{1}{4}$. Therefore, $5 + \frac{1}{4}$, or $5\frac{1}{4}$, is the number. Ans.

89. Rule 7.—*Divide the numerator by the denominator; the quotient will be the whole number; the remainder, if there be any, will be the numerator of the fractional part of which the denominator is the same as the denominator of the improper fraction.*

EXAMPLES FOR PRACTICE.

90. Reduce to whole or mixed numbers :

(a) $1\frac{1}{4}$.

(b) $1\frac{1}{2}$.

(c) $2\frac{1}{8}$.

(d) $1\frac{1}{2}$.

(e) $1\frac{1}{3}$.

(f) $1\frac{1}{4}$.

Ans. $\left\{ \begin{array}{l} (a) 2\frac{1}{4}. \\ (b) 61\frac{1}{2}. \\ (c) 116\frac{1}{8}. \\ (d) 49\frac{1}{2}. \\ (e) 4. \\ (f) 5. \end{array} \right.$

91. A common denominator of two or more fractions is a number which will contain all of the denominators of the fractions without a remainder. The **least common denominator** is the least number that will contain all of the denominators of the fractions without a remainder.

92. To find the least common denominator :

EXAMPLE.—Find the least common denominator of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$.

SOLUTION.—We first place the denominators in a row, separated by commas.

$$\begin{array}{r} 2 \overline{) 4, 3, 9, 16} \\ 2 \overline{) 2, 3, 9, 8} \\ 3 \overline{) 1, 3, 9, 4} \\ 3 \overline{) 1, 1, 3, 4} \\ 4 \overline{) 1, 1, 1, 4} \\ \hline 1, 1, 1, 1 \end{array}$$

$2 \times 2 \times 3 \times 3 \times 4 = 144$, the least common denominator. Ans.

EXPLANATION.—Divide each of them by some prime number which will divide at least two of them without a remainder (if possible), bringing down those denominators to the row

below which will not contain the divisor without a remainder. Dividing each of the numbers by 2, the second row becomes 2, 3, 9, 8, since 2 will not divide 3 and 9 without a remainder. Dividing again by 2, the result is 1, 3, 9, 4. Dividing the third row by 3, the result is 1, 1, 3, 4. So continue until the last row contains only 1's. The product of all the divisors, or $2 \times 2 \times 3 \times 3 \times 4 = 144$, is the least common denominator.

93. EXAMPLE.—Find the least common denominator of $\frac{1}{3}$, $\frac{5}{12}$, $\frac{7}{18}$.

SOLUTION.—

$$\begin{array}{r} 3 \overline{) 9, 12, 18} \\ 3 \overline{) 3, 4, 6} \\ 2 \overline{) 1, 4, 2} \\ 2 \overline{) 1, 2, 1} \\ 1, 1, 1 \end{array}$$

$$3 \times 3 \times 2 \times 2 = 36. \text{ Ans.}$$

94. To reduce two or more fractions to fractions having a common denominator :

EXAMPLE.—Reduce $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{2}$ to fractions having a common denominator.

SOLUTION.—The common denominator is a number which will contain 3, 4, and 2. The least common denominator is 12, because it is the smallest number which can be divided by 3, 4, and 2 without a remainder.

$$\frac{1}{3} = \frac{4}{12}, \frac{1}{4} = \frac{3}{12}, \frac{1}{2} = \frac{6}{12}.$$

Reducing $\frac{1}{3}$, 3 is contained in 12, 4 times. By multiplying both numerator and denominator of $\frac{1}{3}$ by 4, we find

$$\frac{2 \times 4}{3 \times 4} = \frac{8}{12}. \text{ In the same way we find } \frac{1}{4} = \frac{3}{12} \text{ and } \frac{1}{2} = \frac{6}{12}.$$

95. Rule 8.—*Divide the common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

EXAMPLES FOR PRACTICE.

96. Reduce to fractions having a common denominator:

(a) $\frac{1}{2}, \frac{2}{3}, \frac{1}{5}$.	Ans. $\left\{ \begin{array}{l} (a) \frac{6}{30}, \frac{20}{30}, \frac{6}{30}. \\ (b) \frac{9}{36}, \frac{24}{36}, \frac{7}{36}. \\ (c) \frac{7}{28}, \frac{7}{28}, \frac{3}{28}. \\ (d) \frac{24}{120}, \frac{25}{120}, \frac{11}{120}. \\ (e) \frac{16}{160}, \frac{9}{160}, \frac{10}{160}. \\ (f) \frac{11}{110}, \frac{17}{110}, \frac{11}{110}. \end{array} \right.$
(b) $\frac{3}{12}, \frac{1}{4}, \frac{1}{12}$.	
(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{12}$.	
(d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{12}$.	
(e) $\frac{1}{10}, \frac{1}{20}, \frac{1}{30}$.	
(f) $\frac{1}{12}, \frac{1}{15}, \frac{1}{18}$.	

ADDITION OF FRACTIONS.

97. *Fractions can not be added unless they have a common denominator.* We can not add $\frac{2}{4}$ to $\frac{1}{8}$ as they now stand, since the denominators represent parts of different sizes. Fourths can not be added to eighths.

Suppose we divide an apple into 4 equal parts, and then divide 2 of these parts into two equal parts. It is evident that we shall have 2 one-fourths and 4 one-eighths. Now if we add these parts the result is $2 + 4 = 6$ something. But what is this something? It is not fourths, for six fourths are $1\frac{1}{2}$, and we had only 1 apple to begin with; neither is it eighths, for six eighths are $\frac{3}{4}$, which is less than 1 apple. By reducing the quarters to eighths, we have $\frac{2}{4} = \frac{4}{8}$, and adding the other 4 eighths, $4 + 4 = 8$ eighths. The result is correct, since $\frac{8}{8} = 1$. Or we can, in this case, reduce the eighths to quarters. Thus, $\frac{4}{8} = \frac{2}{4}$; whence, adding, $2 + 2 = 4$ quarters, a correct result, since $\frac{4}{4} = 1$.

Before adding, fractions should be reduced to a common denominator, preferably the *least* common denominator.

98. **EXAMPLE.**—Find the sum of $\frac{1}{4}$, $\frac{2}{8}$, and $\frac{5}{8}$.

SOLUTION.—The *least common denominator* or the *least number* which will contain all the *denominators* is 8.

$$\frac{1}{4} = \frac{2}{8}, \frac{2}{8} = \frac{2}{8}, \text{ and } \frac{5}{8} = \frac{5}{8}.$$

$$\frac{2}{8} + \frac{2}{8} + \frac{5}{8} = \frac{2+2+5}{8} = \frac{9}{8} = 1\frac{1}{8}. \quad \text{Ans.}$$

EXPLANATION.—As the *denominator* tells or indicates the names of the *parts*, the *numerators* only are added to obtain the total number of *parts* indicated by the *denominator*.

99. **EXAMPLE.**—What is the sum of $12\frac{3}{4}$, $14\frac{1}{8}$, and $7\frac{5}{8}$?

SOLUTION.—The least common denominator in this case is 16.

$$\begin{array}{r} 12\frac{3}{4} = 12\frac{6}{8} \\ 14\frac{1}{8} = 14\frac{2}{8} \\ 7\frac{5}{8} = 7\frac{5}{8} \\ \hline \text{sum } 33 + \frac{13}{8} = 33 + 1\frac{1}{8} = 34\frac{1}{8}. \quad \text{Ans.} \end{array}$$

The sum of the fractions = $\frac{13}{8}$ or $1\frac{1}{8}$, which added to the sum of the whole numbers = $34\frac{1}{8}$.

EXAMPLE.—What is the sum of 17 , $13\frac{5}{8}$, $\frac{9}{8}$, and $3\frac{1}{2}$?

SOLUTION.—The least common denominator is 32. $13\frac{5}{8} = 13\frac{5}{8}$, $3\frac{1}{2} = 3\frac{4}{8}$.

$$\begin{array}{r} 17 \\ 13\frac{5}{8} \\ \frac{9}{8} \\ 3\frac{4}{8} \\ \hline \text{sum } 33\frac{11}{8}. \text{ Ans.} \end{array}$$

100. Rule 9.—(a) Reduce the given fractions to fractions having the least common denominator, and write the sum of the numerators over the common denominator.

(b) When there are mixed numbers and whole numbers, add the fractions first, and if their sum is an improper fraction, reduce it to a mixed number and add the whole number with the other whole numbers.

EXAMPLES FOR PRACTICE.

101. Find the sum of

(a) $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{8}$.	Ans. {	(a) $1\frac{7}{8}$.
(b) $\frac{3}{4}$, $\frac{5}{16}$, $2\frac{1}{4}$.		(b) $1\frac{13}{16}$.
(c) $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$.		(c) $1\frac{11}{16}$.
(d) $\frac{5}{8}$, $\frac{1}{2}$, $1\frac{3}{8}$.		(d) $1\frac{15}{8}$.
(e) $1\frac{10}{16}$, $\frac{5}{16}$, $2\frac{3}{8}$.		(e) $1\frac{23}{8}$.
(f) $2\frac{3}{8}$, $1\frac{1}{2}$, $1\frac{1}{4}$.		(f) $1\frac{11}{4}$.
(g) $1\frac{1}{16}$, $\frac{7}{16}$, $1\frac{1}{4}$.		(g) $1\frac{7}{8}$.
(h) $\frac{3}{4}$, $1\frac{1}{4}$, $\frac{3}{4}$.		(h) 1.

1. The weights of a number of castings were $412\frac{1}{4}$ lb., $270\frac{1}{2}$ lb., $1,020$ lb., $75\frac{1}{2}$ lb., and $68\frac{1}{2}$ lb. What was their total weight?

Ans. 1,847 lb.

2. Four bolts are required, $2\frac{1}{4}$, $1\frac{1}{2}$, $2\frac{5}{8}$, and $1\frac{1}{2}$ inches long. How long a piece of iron will be required to cut them from, allowing $\frac{3}{8}$ of an inch altogether for cutting off and finishing the ends? Ans. $9\frac{3}{8}$ in.

SUBTRACTION OF FRACTIONS.

102. Fractions can not be subtracted without first reducing them to a common denominator. This can be shown in the same manner as in the case of addition of fractions.

EXAMPLE.—Subtract $\frac{6}{16}$ from $1\frac{13}{16}$.

SOLUTION.—The common denominator is 16.

$$1\frac{13}{16} - \frac{6}{16} = \frac{13 - 6}{16} = 1\frac{7}{16}. \text{ Ans.}$$

103. EXAMPLE.—From 7 take $\frac{1}{4}$.

SOLUTION.— $1 = \frac{4}{4}$; therefore, since $7 = 6 + 1$ we see that $7 = 6 + \frac{4}{4}$, so that $6\frac{4}{4} - \frac{1}{4} = 6\frac{3}{4}$. Ans.

104. EXAMPLE.—What is the difference between $17\frac{9}{16}$ and $9\frac{1}{4}$?

SOLUTION.—The common denominator of the fractions is 32. $17\frac{9}{16}$ is $17\frac{18}{32}$.

$$\begin{array}{r} \text{minuend } 17\frac{18}{32} \\ \text{subtrahend } 9\frac{8}{32} \\ \hline \text{difference } 8\frac{10}{32} \text{ Ans.} \end{array}$$

105. EXAMPLE.—From $9\frac{1}{4}$ take $4\frac{7}{8}$.

SOLUTION.—The common denominator of the fractions is 16. $9\frac{1}{4}$ is $9\frac{4}{16}$.

$$\begin{array}{r} \text{minuend } 9\frac{4}{16} \quad 8\frac{10}{16} \\ \text{subtrahend } 4\frac{14}{16} \text{ or } 4\frac{7}{8} \\ \hline \text{difference } 4\frac{10}{16} \quad 4\frac{5}{8} \text{ Ans.} \end{array}$$

EXPLANATION.—As the *fraction* in the *subtrahend* is greater than the *fraction* in the *minuend*, it *can not* be subtracted; therefore, *borrow* 1, or $\frac{16}{16}$, from the 9 in the *minuend* and *add* it to the $\frac{4}{16}$; $\frac{4}{16} + \frac{16}{16} = \frac{20}{16}$. $\frac{14}{16}$ from $\frac{20}{16} = \frac{6}{16}$. Since 1 was *borrowed* from 9, 8 *remains*; 4 from 8 = 4; $4 + \frac{6}{16} = 4\frac{3}{8}$.

106. EXAMPLE.—From 9 take $8\frac{3}{8}$.

$$\begin{array}{r} \text{minuend } 9 \quad 8\frac{8}{8} \\ \text{subtrahend } 8\frac{3}{8} \text{ or } 8\frac{3}{8} \\ \hline \text{difference } 1\frac{5}{8} \quad 1\frac{5}{8} \text{ Ans.} \end{array}$$

EXPLANATION.—As there is no *fraction* in the *minuend* from which to take the *fraction* in the *subtrahend*, *borrow* 1, or $\frac{8}{8}$, from 9. $\frac{8}{8}$ from $\frac{8}{8} = \frac{0}{8}$. Since 1 was *borrowed* from 9, only 8 is left. 8 from 8 = 0.

107. Rule 10.—(a) Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.

(b) When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.

(c) When the fraction in the subtrahend is greater than the fraction in the minuend, borrow 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.

(d) When the minuend is a whole number, borrow 1; reduce it to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.

EXAMPLES FOR PRACTICE.

108. Subtract

(a) $\frac{1}{2}$ from $\frac{1}{3}$.

(b) $\frac{1}{4}$ from $\frac{1}{5}$.

(c) $\frac{1}{6}$ from $\frac{1}{7}$.

(d) $\frac{1}{8}$ from $\frac{1}{9}$.

(e) $\frac{1}{10}$ from $\frac{1}{11}$.

(f) $13\frac{1}{2}$ from $30\frac{1}{2}$.

(g) $12\frac{1}{2}$ from 27.

(h) $5\frac{1}{2}$ from 30.

Ans. $\left\{ \begin{array}{l} (a) \frac{1}{6} \\ (b) \frac{1}{20} \\ (c) \frac{1}{42} \\ (d) \frac{1}{72} \\ (e) \frac{1}{110} \\ (f) 17\frac{1}{2} \\ (g) 14\frac{1}{2} \\ (h) 24\frac{1}{2} \end{array} \right.$

1. An engineer found that he had on hand $48\frac{1}{2}$ gallons of cylinder oil. During the following week he used $\frac{2}{3}$ of a gallon a day for three days, $\frac{1}{3}$ of a gallon on the fourth day, $\frac{1}{6}$ of a gallon on the fifth day, and $\frac{1}{6}$ of a gallon on the sixth day. How much oil remained at the end of the week?

Ans. $43\frac{1}{6}$ gallons.

2. The main line shaft of a manufacturing plant is run by an engine and water wheel. A test of the plant showed that the engine was capable of developing $251\frac{1}{2}$ H. P. (horsepower), and the water wheel, under full gate, $67\frac{1}{2}$ H. P. It was also found that the machinery consumed $210\frac{1}{2}$ H. P., and the friction of the shafting and belting was $32\frac{1}{2}$ H. P. How much power remained unused?

Ans. $76\frac{1}{2}$ H. P.

MULTIPLICATION OF FRACTIONS.

109. In *multiplication* of fractions it is not necessary to *reduce the fractions* to fractions having a *common denominator*.

110. Multiplying the *numerator* or *dividing the denominator multiplies* the fraction.

EXAMPLE.—Multiply $\frac{3}{4} \times 4$.

SOLUTION.—

$$\frac{3}{4} \times 4 = \frac{3 \times 4}{4} = 1^3 = 3, \text{ Ans}$$

$$\text{or, } \frac{3}{4} \times 4 = \frac{3}{4 + 4} = 1 = 3. \text{ Ans.}$$

The word “of” in multiplication of fractions means the same as $\times$, or times. Thus,

$$\frac{3}{4} \text{ of } 4 = \frac{3}{4} \times 4 = 3.$$

$$\frac{1}{8} \text{ of } \frac{5}{16} = \frac{1}{8} \times \frac{5}{16} = \frac{5}{128}.$$

EXAMPLE.—Multiply $\frac{3}{8}$ by 2.

$$\text{SOLUTION.—} \quad \frac{3}{8} \times 2 = \frac{3 \times 2}{8} = \frac{6}{8} = \frac{3}{4}, \text{ Ans.}$$

$$\text{or, } \frac{3}{8} \times 2 = \frac{3}{8 + 2} = \frac{3}{4}. \text{ Ans.}$$

111. EXAMPLE.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

$$\text{SOLUTION.—} \quad \frac{4}{16} \times \frac{7}{8} = \frac{4 \times 7}{16 \times 8} = \frac{28}{128} = \frac{7}{32}.$$

$$\text{or, by cancelation, } \frac{4 \times 7}{16 \times 8} = \frac{7}{4 \times 8} = \frac{7}{32}. \text{ Ans.}$$

112. EXAMPLE.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{2}$?

$$\text{SOLUTION.—} \quad \frac{4 \times 3 \times 16}{8 \times 4 \times \frac{2}{2}} = \frac{3}{8 \times 2} = \frac{3}{16}. \text{ Ans.}$$

113. EXAMPLE.—What is the product of $9\frac{1}{4}$ and $5\frac{5}{8}$?

$$\text{SOLUTION.—} \quad 9\frac{1}{4} = \frac{37}{4}; \quad 5\frac{5}{8} = \frac{45}{8}.$$

$$\frac{37}{4} \times \frac{45}{8} = \frac{39 \times 45}{4 \times 8} = \frac{1755}{32} = 54\frac{27}{32}. \text{ Ans.}$$

114. EXAMPLE.—Multiply $15\frac{7}{8}$ by 3.

$$\text{SOLUTION.—} \quad \begin{array}{r} 15\frac{7}{8} \\ 3 \text{ or } 3 \\ \hline 47\frac{1}{8} \end{array} \quad \begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 45 + \frac{21}{8} = 45 + 2\frac{5}{8} = 47\frac{5}{8}. \end{array} \text{ Ans.}$$

115. Rule 11.—(a) *Divide the product of the numerators by the product of the denominators. All factors common to the numerators and denominators should first be cast out by cancelation.*

(b) To multiply one mixed number by another, reduce them both to improper fractions.

(c) To multiply a mixed number by a whole number, first multiply the fractional part by the multiplier, and if the product is an improper fraction, reduce it to a mixed number and add the whole number part to the product of the multiplier and whole number.

EXAMPLES FOR PRACTICE.

116. Find the product of

(a) $7 \times \frac{3}{15}$.	Ans. {	(a) $1\frac{2}{15}$.
(b) $14 \times \frac{5}{18}$.		(b) $4\frac{5}{9}$.
(c) $\frac{3}{8} \times \frac{5}{14}$.		(c) $\frac{15}{112}$.
(d) $\frac{1}{3} \times 4$.		(d) $2\frac{1}{3}$.
(e) $\frac{1}{3} \times 7$.		(e) $7\frac{1}{3}$.
(f) $17\frac{1}{11} \times 7$.		(f) 125.
(g) $\frac{1}{2} \times 32$.		(g) 15.
(h) $\frac{1}{8} \times 14$.		(h) $7\frac{1}{2}$.

1. A single belt can transmit $107\frac{1}{2}$ H. P., but as it is desired to use more power, a double belt of the same width is substituted for it. Supposing the double belt to be capable of transmitting $\frac{1}{7}$ as much power as the single belt, how many H. P. can be used after the change?

Ans. $153\frac{1}{7}$ H. P.

2. What is the weight of $2\frac{5}{8}$ miles of copper wire weighing $5\frac{1}{2}$ pounds per 100 feet? There are 5,280 feet in a mile.

Ans. $796\frac{1}{8}$ pounds.

3. The grate of a steam boiler contains $20\frac{1}{2}$ square feet. If the boiler burns $8\frac{3}{10}$ pounds of coal an hour per square foot of grate area, and can evaporate $7\frac{1}{2}$ pounds of water an hour per pound of coal burned, how many pounds of water are evaporated by the boiler in 1 hour?

Ans. $1,276\frac{1}{2}$ pounds.

DIVISION OF FRACTIONS.

117. In *division* of fractions it is not necessary to *reduce the fractions* to fractions having a *common denominator*.

118. Dividing the *numerator* or *multiplying the denominator* divides the fraction.

EXAMPLE.—Divide $\frac{3}{8}$ by 3.

SOLUTION.—When *dividing* the *numerator*, we have

$$\frac{3}{8} \div 3 = \frac{6}{8} \div 3 = \frac{1}{4}. \quad \text{Ans.}$$

When *multiplying* the *denominator*, we have

$$\frac{3}{8} \div 3 = \frac{3}{8 \times 3} = \frac{1}{8} = \frac{1}{4}. \quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{1}{16}$ by 2.

SOLUTION.— $\frac{1}{16} \div 2 = \frac{3}{16} \times 2 = \frac{1}{8}. \quad \text{Ans.}$

EXAMPLE.—Divide $\frac{1}{32}$ by 7.

SOLUTION.— $\frac{1}{32} \div 7 = \frac{14}{32} \div 7 = \frac{1}{16} = \frac{1}{16}. \quad \text{Ans.}$

119. To *invert* a fraction is to *turn it upside down*; that is, make the *numerator* and *denominator change places*. Invert $\frac{3}{4}$ and it becomes $\frac{4}{3}$.

120. EXAMPLE.—Divide $\frac{9}{16}$ by $\frac{1}{8}$.

SOLUTION.—1. The fraction $\frac{1}{8}$ is contained in $\frac{9}{16}$, 3 times, for the *denominators* are the same, and one *numerator* is contained in the other 3 times. 2. If we now *invert* the *divisor*, $\frac{1}{8}$, and *multiply*, the solution is

$$\frac{9}{16} \times \frac{16}{3} = \frac{9 \times 16}{16 \times 3} = 3. \quad \text{Ans.}$$

This gives the *same quotient* as in the first case.

121. EXAMPLE.—Divide $\frac{3}{8}$ by $\frac{1}{4}$.

SOLUTION.—We can not divide $\frac{3}{8}$ by $\frac{1}{4}$, as in the first case above, for the *denominators* are *not* the same; therefore, we must solve as in the second case.

$$\frac{3}{8} \div \frac{1}{4} = \frac{3}{8} \times \frac{4}{1} = \frac{3 \times 4}{8 \times 1} = \frac{3}{2} \text{ or } 1\frac{1}{2}. \quad \text{Ans.}$$

122. EXAMPLE.—Divide 5 by $\frac{1}{10}$.

SOLUTION.— $\frac{1}{10}$ inverted becomes $\frac{10}{1}$,

$$5 \times \frac{16}{10} = \frac{5 \times 16}{10} = 8. \quad \text{Ans.}$$

123. EXAMPLE.—How many times is $3\frac{1}{2}$ contained in $7\frac{7}{8}$?

SOLUTION.—

$$3\frac{1}{2} = \frac{7}{2}; \quad 7\frac{7}{8} = 1\frac{7}{8}.$$

$\frac{7}{2}$ inverted becomes $\frac{2}{7}$.

$$\frac{119}{16} \times \frac{2}{7} = \frac{119 \times 2}{16 \times 7} = \frac{119}{60} = 1\frac{19}{60}. \quad \text{Ans.}$$

124. Rule 12.—*Invert the divisor and proceed as in multiplication.*

125. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ shows that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\frac{9}{\frac{8}{3}}$ means that 9 is to be divided by $\frac{8}{3}$; $\frac{3 \times 7}{\frac{8+4}{16}}$ means that

3×7 is to be divided by the value of $\frac{8+14}{16}$.

$\frac{\frac{1}{4}}{\frac{8}{3}}$ is the same as $\frac{1}{4} \div \frac{8}{3}$.

It will be noticed that there is a heavy line between the 9 and the $\frac{8}{3}$. This is necessary, since otherwise there would be nothing to show whether 9 is to be divided by $\frac{8}{3}$, or $\frac{9}{8}$ by 8. Whenever a heavy line is used, as in the above case, it indicates that *all above the line* is to be divided by *all below it*.

EXAMPLES FOR PRACTICE.

126. Divide

(a) 15 by $6\frac{3}{4}$.

(b) 30 by $\frac{5}{8}$.

(c) 172 by $\frac{4}{5}$.

(d) $\frac{1}{2}$ by $1\frac{1}{16}$.

(e) $1\frac{3}{4}$ by $14\frac{3}{8}$.

(f) $1\frac{1}{2}$ by $17\frac{1}{2}$.

(g) $\frac{1}{2}$ by $1\frac{1}{4}$.

(h) $1\frac{3}{8}$ by $72\frac{1}{2}$.

Ans. $\left\{ \begin{array}{l} (a) 2\frac{1}{4}. \\ (b) 40. \\ (c) 215. \\ (d) \frac{1}{16}. \\ (e) 1\frac{1}{8}. \\ (f) \frac{1}{112}. \\ (g) \frac{1}{8}. \\ (h) \frac{1}{144}. \end{array} \right.$

1. A $\frac{3}{8}$ -inch boiler plate containing 24 square feet of surface weighs 362 $\frac{1}{10}$ pounds. What is its weight per square foot? Ans. 15 $\frac{1}{10}$ pounds.

2. A certain boiler has $927\frac{1}{2}$ square feet of heating surface, which is equal to 35 times the area of the grate. What is the area of the grate in square feet? Ans. $26\frac{1}{2}$ square feet.

3. If the distance around the rim of a locomotive drive wheel is $13\frac{1}{2}$ feet, how many revolutions will the wheel make in traveling 682 feet? Ans. $52\frac{20}{27}$ revolutions.

127. Whenever an expression like one of the three following ones is obtained, it may always be simplified by transposing the denominator from *above* to *below* the line, or from *below* to *above*, as the case may be, taking care, however, to indicate that the denominator when so transferred is a multiplier.

1. $\frac{\frac{3}{2}}{9} = \frac{3}{9 \times 4} = \frac{3}{36} = \frac{1}{12}$; for, regarding the fraction above the heavy line as the numerator of a fraction whose denominator is 9, $\frac{\frac{3}{2} \times 4}{9 \times 4} = \frac{3}{9 \times 4}$, as before.

2. $\frac{9}{\frac{3}{4}} = \frac{9 \times 4}{3} = 12$. The proof is the same as in the first case.

3. $\frac{\frac{5}{3}}{\frac{3}{4}} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$; for, regarding $\frac{5}{3}$ as the numerator of a fraction whose denominator is $\frac{3}{4}$, $\frac{\frac{5}{3} \times 9}{\frac{3}{4} \times 9} = \frac{5}{3 \times 9}$; and $\frac{\frac{5}{3 \times 9} \times 4}{\frac{3}{4} \times 4} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$, as above.

This principle may be used to great advantage in cases like $\frac{\frac{1}{4} \times 310 \times \frac{27}{31} \times 72}{40 \times 4\frac{1}{2} \times 5\frac{1}{6}}$. Reducing the mixed numbers to fractions, the expression becomes $\frac{\frac{1}{4} \times 310 \times \frac{27}{31} \times 72}{40 \times \frac{9}{2} \times \frac{31}{6}}$. Now transferring the denominators of the fractions and canceling,

$$\frac{1 \times 310 \times 27 \times 72 \times 2 \times 6}{40 \times 9 \times 31 \times 4 \times 12} = \frac{1 \times \overset{10}{\cancel{310}} \times \overset{3}{\cancel{27}} \times \overset{6}{\cancel{72}} \times 2 \times \overset{3}{\cancel{6}}}{\underset{4}{\cancel{40}} \times 9 \times \cancel{31} \times \underset{2}{\cancel{4}} \times \cancel{12}} = \frac{27}{2} = 13\frac{1}{2}.$$

Greater exactness in results can usually be obtained by using this principle than can be obtained by reducing the fractions to decimals. The principle, however, should not be employed *if a sign of addition or subtraction occurs either above or below the dividing line.*

DECIMALS.

128. **Decimals** are *tenth* fractions ; that is, the parts of a unit are expressed on the scale of ten, as *tenths*, *hundredths*, *thousandths*, etc.

129. The *denominator*, which is always ten or a multiple of ten, as 10, 100, 1,000, etc., is *not* expressed as it would be in common fractions, by writing it under the *numerator*, with a line between them; as, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$. The denominator is always understood, the numerator consisting of the figures on the right of the *unit* figure. In order to distinguish the unit figure, a period (.), called the **decimal point**, is placed between the unit figure and the next figure on the right. The decimal point may be regarded in two ways : first, as indicating that the number on the right is the numerator of a fraction whose denominator is 10, 100, 1,000, etc. ; and, second, as a part of the Arabic system of notation, each figure on the right being 10 times as large as the next succeeding figure, and 10 times as small as the next preceding figure, serving merely to point out the unit figure.

130. The *reading* of a *decimal number* depends upon the *number of decimal places* in it, or the *number of figures* to the *right* of the unit figure.

The first figure to the right of the unit figure expresses *tenths*.

The second figure to the right of the unit figure expresses *hundredths*.

The third figure to the right of the unit figure expresses *thousandths*.

The fourth figure to the right of the unit figure expresses *ten-thousandths*.

The fifth figure to the right of the unit figure expresses *hundred-thousandths*.

The sixth figure to the right of the unit figure expresses *millionths*.

Thus:

$$\begin{aligned} .3 &= \frac{3}{10} = 3 \text{ tenths.} \\ .03 &= \frac{3}{100} = 3 \text{ hundredths.} \\ .003 &= \frac{3}{1000} = 3 \text{ thousandths.} \\ .0003 &= \frac{3}{10000} = 3 \text{ ten-thousandths.} \\ .00003 &= \frac{3}{100000} = 3 \text{ hundred-thousandths.} \\ .000003 &= \frac{3}{1000000} = 3 \text{ millionths.} \end{aligned}$$

The first figure to the right of the unit figure is called the *first decimal place*; the second figure, the *second decimal place*, etc. We see in the above that the *number of decimal places* in a decimal equals the *number of ciphers* to the *right* of the figure 1 in the *denominator* of its *equivalent fraction*. This fact kept in mind will be of much assistance in reading and writing decimals.

Whatever may be written to the *left* of a *decimal point* is a whole number. The decimal point affects only the figures to its *right*.

When a *whole number* and *decimal* are written together, the expression is a *mixed number*. Thus, 8.12 and 17.25 are mixed numbers.

The relation of decimals and whole numbers to each other is clearly shown by the following table :

9	hundreds of millions.	9	hundred-thousandths.
8	tens of millions.	8	millionths.
7	millions.	7	ten-millionths.
6	hundreds of thousands.	6	hundred-thousandths.
5	tens of thousands.	5	millionths.
4	thousands.	4	ten-thousandths.
3	hundreds.	3	hundred-thousandths.
2	tens.	2	millionths.
1	units.	1	ten-millionths.
.	decimal point.		
2	tenths.		
3	hundredths.		
4	thousandths.		
5	ten-thousandths.		
6	hundred-thousandths.		
7	millionths.		
8	ten-millionths.		
9	hundred-millionths.		

The figures to the *left* of the *decimal point* represent *whole numbers*; those to the *right* are *decimals*.

In *both* the decimals and whole numbers, the *units* place is made the *starting point* of notation and numeration. The

decimals decrease on the scale of *ten* to the *right*, and the *whole numbers increase* on the scale of *ten* to the *left*. The *first* figure to the *left* of units is *tens*, and the *first* figure to the *right* of units is *tenths*. The *second* figure to the *left* of units is *hundreds*, and the *second* figure to the *right* is *hundredths*. The *third* figure to the *left* is *thousands*, and the *third* to the *right* is *thousandths*, and so on ; the *whole numbers* on the *left* and the *decimals* on the *right*. The figures equally distant from units place correspond in name. The *decimals* have the ending *ths*, which distinguishes them from *whole numbers*. The following is the numeration of the number in the above table : Nine hundred eighty-seven million, six hundred fifty-four thousand, three hundred twenty-one, and twenty-three million, four hundred fifty-six thousand, seven hundred eighty-nine hundred millionths.

The *decimals* increase to the *left* on a scale of *ten*, the same as *whole numbers*, for if you begin at *4-thousandths*, in the table, you see that the next figure is *hundredths*, which is ten times as great, and the next *tenths*, or ten times the *hundredths*, and so on through both decimals and whole numbers.

131. *Annexing* or *taking away* a *cipher* at the *right* of a *decimal* does *not* affect its value.

.5 is $\frac{5}{10}$; .50 is $\frac{50}{100}$, but $\frac{5}{10} = \frac{50}{100}$; therefore, .5 = .50.

132. *Inserting* a *cipher* between a *decimal* and the *decimal point* *divides* the decimal by 10.

$$.5 = \frac{5}{10}; \frac{5}{10} \div 10 = \frac{5}{100} = .05.$$

133. *Taking away* a *cipher* from the *left* of a *decimal* *multiplies* the decimal by 10.

$$.05 = \frac{5}{100}; \frac{5}{100} \times 10 = \frac{5}{10} = .5.$$

ADDITION OF DECIMALS.

134. The only respect in which *addition* of *decimals* differs from *addition* of *whole numbers* is in the placing of the numbers to be added.

Whole numbers begin at units and increase on the scale of 10 to the left. Decimals decrease on the scale of 10 to

the right. Whole numbers are to the left of the decimal point and decimals are to the right of it. In whole numbers the *right hand* side of a column of figures to be added must be in line, and in decimals the *left hand* side must be in line, which brings the decimal points directly under each other.

<i>whole numbers</i>	<i>decimals</i>	<i>mixed numbers</i>
342	.342	342.032
4234	.4234	4234.5
26	.26	26.6782
3	.03	3.06
<u>sum 4605</u> Ans.	<u>sum 1.0554</u> Ans.	<u>sum 4606.2702</u> Ans.

135. EXAMPLE.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

SOLUTION.—

242.
.36
118.725
1.005
6.
100.1
<u>sum 468.190</u> Ans.

136. Rule 13.—Place the numbers to be added so that the decimal points will be directly under each other. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.

EXAMPLES FOR PRACTICE.

137. Find the sum of

(a) .2148, .105, 2.8042, and 1.1417.	Ans. {	(a) 3.7652.
(b) 788.5, 21.478, .2101, and .7816.		(b) 805.9647.
(c) 21.781, 188.72, 41.8788, .72, and 1.413.		(c) 204.5078.
(d) .3724, 104.15, 21.417, and 100.042.		(d) 225.9814.
(e) 200.172, 14.105, 12.1465, .705, and 7.2.		(e) 234.3285.
(f) 1,427.16, .244, .32, .032, and 10.0041.		(f) 1,437.7601.
(g) 2,478.1, 41.65, .7248, 104.067, and 21.073.		(g) 2,640.6143.
(h) 4,107.2, .00875, 21.716, 410.072, and .0345.		(h) 4,539.02625.

1. The estimated weights of the parts of a return tubular boiler were as follows: Shell, 3,626 lb.; tubes, 3,564.5 lb.; manhole cover, ring, and yoke, 270.84 lb.; stays, etc., 1,089.4 lb.; steam nozzles,

236.07 lb.; handhole covers and yokes, 120.25 lb.; feed pipe, 34.75 lb.; boiler supports, 350.6 lb. What was the total estimated weight of the boiler? Ans. 9,291.91 lb.

2. A bill for engine room supplies had the following items: 1 waste can, \$8.30; 20 feet of 4-inch belting, \$11.20; 1 pipe wrench, \$1.65; 12 pounds of waste, \$0.84; 5 gallons of cylinder oil, \$8.75; 20 gallons of machine oil, \$24. How much did the bill amount to? Ans. \$54.74.

SUBTRACTION OF DECIMALS.

138. For the same reason as in addition of decimals, the *left hand* figures of *decimal numbers* are placed in line and the *decimal points* under each other.

EXAMPLE.—Subtract .132 from .3063.

SOLUTION. —
$$\begin{array}{r} \text{minuend } .3063 \\ \text{subtrahend } .132 \\ \hline \text{difference } .1743 \end{array}$$
 Ans.

139. EXAMPLE.—What is the difference between 7.895 and .725?

SOLUTION.—
$$\begin{array}{r} \text{minuend } 7.895 \\ \text{subtrahend } .725 \\ \hline \text{difference } 7.170 \text{ or } 7.17 \end{array}$$
 Ans.

140. EXAMPLE.—Subtract .625 from 11.

SOLUTION.—
$$\begin{array}{r} \text{minuend } 11.000 \\ \text{subtrahend } .625 \\ \hline \text{difference } 10.375 \end{array}$$

141. Rule 14.—Place the subtrahend under the minuend, so that the decimal points will be directly under each other. Subtract as in whole numbers, and place the decimal point in the remainder, directly under the decimal points above.

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them and subtract as before.

EXAMPLES FOR PRACTICE.

142. From

- (a) 407.385 take 235.0004.
 (b) 22.718 take 1.7042.
 (c) 1,368.17 take 13.6817.
 (d) 70.00017 take 7.000017.

Ans. $\left\{ \begin{array}{l} (a) 172.3846. \\ (b) 21.0138. \\ (c) 1,354.4883. \\ (d) 63.000153. \end{array} \right.$

- (*e*) 630.630 take .6304.
 (*f*) 421.73 take 217.162.
 (*g*) 1.000014 take .00001.
 (*h*) .783652 take .542814.

$$\text{Ans. } \left\{ \begin{array}{l} (*e*) \ 629.9996. \\ (*f*) \ 204.568. \\ (*g*) \ 1.000004. \\ (*h*) \ .241338. \end{array} \right.$$

1. If the temperature of steam at 5 pounds pressure is 227.964 degrees, and at 100 pounds pressure 337.874 degrees, how many degrees warmer is the steam at the higher pressure? Ans. 109.91 degrees.

2. The outside diameter of $2\frac{1}{4}$ -inch wrought-iron pipe is 2.87 inches and the inside diameter 2.46 inches. How thick is the pipe? Ans. $.41 \div 2 = .205$ inch.

3. In a cistern that will hold 326.5 barrels of water there are 178.625 barrels. How much does it lack of being full? Ans. 147.875 barrels.

4. A wrought-iron rod is 2.53 inches in diameter. What must be the thickness of metal turned off so that the rod will be 2.495 inches in diameter? Ans. $.035 \div 2 = .0175$.

MULTIPLICATION OF DECIMALS.

143. In multiplication of decimals, we do not place the *decimal points* directly *under* each other as in addition and subtraction. We pay *no* attention for the time being to the decimal points. Place the multiplier under the multiplicand, so that the *right-hand* figure of the one is under the *right-hand* figure of the other, and proceed exactly as in multiplication of whole numbers. After multiplying, *count the number of decimal places in both multiplicand and multiplier, and point off the same number in the product.*

EXAMPLE.—Multiply .825 by 13.

$$\begin{array}{r} \text{SOLUTION.—} \quad \text{multiplicand} \quad .825 \\ \quad \text{multiplier} \quad \quad \quad 13 \\ \hline \quad \quad \quad 2475 \\ \quad \quad \quad 825 \\ \hline \text{product} \quad 10.725 \quad \text{Ans.} \end{array}$$

In this example there are three decimal places in the multiplicand and none in the multiplier; therefore, 3 decimal places are pointed off in the product.

144. EXAMPLE.—What is the product of 426 and the decimal .005?

$$\begin{array}{r} \text{SOLUTION.—} \quad \text{multiplicand} \quad 426 \\ \quad \text{multiplier} \quad .005 \\ \hline \text{product} \quad 2.130 \text{ or } 2.13 \text{ Ans.} \end{array}$$

In this example there are 3 decimal places in the multiplier and none in the multiplicand; therefore, 3 decimal places are pointed off in the product.

145. It is *not* necessary to multiply by the ciphers on the *left* of a *decimal*; they merely determine the number of decimal places. Ciphers to the *right* of a decimal should be removed, as they only make more figures to deal with, and do not change the value.

146. EXAMPLE.—Multiply 1.205 by 1.15.

$$\begin{array}{r} \text{SOLUTION.—} \quad \text{multiplicand} \quad 1.205 \\ \quad \text{multiplier} \quad 1.15 \\ \hline \quad \quad \quad 6025 \\ \quad \quad 1205 \\ \quad 1205 \\ \hline \text{product} \quad 1.38575 \text{ Ans.} \end{array}$$

In this example there are 3 decimal places in the multiplicand, and 2 in the multiplier, therefore $3 + 2$, or 5, decimal places must be pointed off in the product.

147. EXAMPLE.—Multiply .232 by .001.

$$\begin{array}{r} \text{SOLUTION.—} \quad \text{multiplicand} \quad .232 \\ \quad \text{multiplier} \quad .001 \\ \hline \text{product} \quad .000232 \text{ Ans.} \end{array}$$

In this example we multiply the multiplicand by the digit in the multiplier, which makes 232 in the product, but since there are 3 decimal places in each, the multiplier and the multiplicand, we must prefix 3 ciphers to the 232, to make $3 + 3$, or 6, decimal places in the product.

148. Rule 15.—Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply as in whole numbers, and in the product point off as

many decimal places as there are decimal places in both multiplier and multiplicand, prefixing ciphers if necessary.

EXAMPLES FOR PRACTICE.

149. Find the product of

- (a) $.000492 \times 4.1418$.
- (b) $4,003.2 \times 1.2$.
- (c) 78.6531×1.03 .
- (d) $.3685 \times .042$.
- (e) $178,352 \times .01$.
- (f) $.00045 \times .0045$.
- (g) $.714 \times .00002$.
- (h) $.00004 \times .008$.

$$\text{Ans. } \left\{ \begin{array}{l} (a) .0020377656. \\ (b) 4,803.84. \\ (c) 81.012693. \\ (d) .015477. \\ (e) 1,783.52. \\ (f) .000002025. \\ (g) .00001428. \\ (h) .0000032. \end{array} \right.$$

1. The stroke of an engine was found by measurement to be 2.987 feet. How many feet will the cross-head pass over in 600 revolutions?

Ans. 3,584.4 feet.

2. If a steam pump delivers 2.39 gallons of water per stroke and runs at 51 strokes a minute, how many gallons of water would it pump in $58\frac{1}{2}$ minutes?

Ans. 7,130.565 gallons.

3. Wishing to obtain the weight of a connecting-rod from a drawing, it was calculated that the rod contained 294.8 cubic inches of wrought iron, 68.5 cubic inches of brass, and 10.4 cubic inches of babbitt. Assuming the weight of wrought iron to be .278 pound per cubic inch, of brass .303 pound, and of babbitt .264 pound, what was the weight of the rod?

Ans. 103.94 pounds.

DIVISION OF DECIMALS.

150. In division of decimals we pay *no* attention to the *decimal point* until *after* the *division* is performed. The *number of decimal places* in the *dividend* must *equal* (be made to equal by annexing ciphers) the *number of decimal places* in the *divisor*. Divide *exactly* as in whole numbers. Subtract the *number of decimal places* in the *divisor* from the *number of decimal places* in the *dividend*, and *point off* as many *decimal*

places in the quotient as there are units in the remainder thus found.

EXAMPLE.—Divide .625 by 25.

	<i>divisor</i>	<i>dividend</i>	<i>quotient</i>	
SOLUTION.—	25	) .625	(.025	Ans.
		50		
		125		
		125		
		0		
	<i>remainder</i>			

In this example there are no decimal places in the divisor, and 3 decimal places in the dividend; therefore, there are 3 minus 0, or 3, decimal places in the quotient. One cipher has to be prefixed to the 25, to make the 3 decimal places.

151. EXAMPLE.—Divide 6.035 by .05.

	<i>divisor</i>	<i>dividend</i>	<i>quotient</i>	
SOLUTION.—	.05	) 6.035	(120.7	Ans.
		5		
		10		
		10		
		35		
		35		
	<i>remainder</i>	0		

In this example we divide by 5, as if the cipher were not before it. There is one more decimal place in the dividend than in the divisor; therefore, one decimal place is pointed off in the quotient.

152. EXAMPLE.—Divide .125 by .005.

	<i>divisor</i>	<i>dividend</i>	<i>quotient</i>	
SOLUTION.—	.005	) .125	(25	Ans.
		10		
		25		
		25		
	<i>remainder</i>	0		

In this example there are the same number of decimal places in the dividend as in the divisor; therefore, the quotient has no decimal places, and is a whole number.

153. EXAMPLE.—Divide 326 by .25.

<i>divisor</i>	<i>dividend</i>	<i>quotient</i>
.25	326.00	(1304 Ans.
	25	
	76	
	75	
	100	
	100	
	remainder 0	

In this problem two ciphers were annexed to the dividend, to make the number of decimal places equal to the number in the divisor. The quotient is a whole number.

154. EXAMPLE.—Divide .0025 by 1.25.

1.25	.00250	(.002 Ans.
	250	
	remainder 0	

EXPLANATION.—In this example we are to divide .0025 by 1.25. Consider the dividend as a whole number, or 25 (disregarding the two ciphers at its left, for the present); also, consider the divisor as a whole number, or 125. It is clearly evident that the dividend 25 will not contain the divisor 125; we must, therefore, annex one cipher to the 25, thus making the dividend 250. 125 is contained twice in 250, so we place the figure 2 in the quotient. In pointing off the decimal places in the quotient, it must be remembered that there were only four decimal places in the dividend; but one cipher was annexed, thereby making $4 + 1$, or 5, decimal places. Since there are 5 decimal places in the dividend and 2 decimal places in the divisor, we must point off $5 - 2$, or 3, decimal places in the quotient. In order to point off 3 decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient. It is not necessary to consider the ciphers at the left of a decimal when dividing, except when determining the position of the decimal point in the quotient.

155. Rule 16.—(a) *Place the divisor to the left of the dividend and proceed as in division of whole numbers, and in the*

quotient, point off as many decimal places as the number of decimal places in the dividend exceed those in the divisor, prefixing ciphers to the quotient, if necessary.

(b) *If in dividing one number by another there be a remainder, the remainder can be placed over the divisor, as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still be a remainder, terminate the quotient by the plus sign (+), which shows that it can be carried further.*

156. EXAMPLE.—What is the quotient of 199 divided by 15?

SOLUTION.—

$$\begin{array}{r}
 15 \overline{) 199} \quad (13 + \frac{4}{15} \text{ Ans.} \\
 \underline{45} \\
 49 \\
 \underline{45} \\
 \text{remainder } 4
 \end{array}$$

Or, $15 \overline{) 199.000} (13.266 + \text{Ans.}$

$$\begin{array}{r}
 15 \overline{) 199.000} \\
 \underline{45} \\
 49 \\
 \underline{45} \\
 40 \\
 \underline{30} \\
 100 \\
 \underline{90} \\
 100 \\
 \underline{90} \\
 \text{remainder } 10
 \end{array}$$

$$\begin{array}{l}
 13\frac{4}{15} = 13.266 + \\
 \frac{4}{15} = .266 +
 \end{array}$$

It very frequently happens as in the above example, that the division will never terminate. In such cases, decide to how many decimal places the division is to be carried, and carry the work one place further. If the last figure of the quotient thus obtained is 5 or a greater number, increase the preceding figure by 1, and write after it the minus sign (−), thus indicating that the quotient is not quite as large as indicated; if the figure thus obtained is less than 5, write the plus sign (+) after the quotient, thus indicating that

the number is slightly greater than as indicated. In the last example, had it been desired to obtain the answer correct to four decimal places, the work would have been carried to five places, obtaining 13.2666, and the answer would have been given as 13.2667—. This remark applies to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it is desired to retain three decimal places in the number .2471253, it would be expressed as .247+; if it was desired to retain five decimal places, it would be expressed as .24713—. Both the + and — signs are frequently omitted; they are seldom used outside of arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not quite exact.

EXAMPLES FOR PRACTICE.

157. Divide

- (a) 101.6688 by 2.36.
- (b) 187.12264 by 123.107.
- (c) .08 by .008.
- (d) .0008 by 3.75.
- (e) .0144 by .024.
- (f) .00875 by 1.25.
- (g) .004 by 400.
- (h) .4 by .008.

$$\text{Ans. } \left\{ \begin{array}{l} (a) 43.08. \\ (b) 1.52. \\ (c) 10. \\ (d) .00008. \\ (e) .6. \\ (f) .003. \\ (g) .00001. \\ (h) 50. \end{array} \right.$$

1. In a steam engine test of an hour's duration, the horsepower developed was found to be as follows, at 10-minute intervals: 25.73, 25.64, 26.13, 25.08, 24.20, 26.7, 26.34. What was the average horsepower?
Ans. 25.6886, average.

NOTE.—Add the different horsepowers together and divide by the number of tests, or 7.

2. There are 31.5 gallons in a barrel. How many barrels are there in 2,787.75 gallons?
Ans. 88.5 barrels.

3. A car load of 18.75 tons of coal cost \$60.75. How much was it worth per ton?
Ans. \$3.24 per ton.

4. A keg of $\frac{3}{4}$ by $1\frac{1}{2}$ -inch boiler rivets weighs 100 pounds and contains 595 rivets. What is the weight of one of the rivets?
Ans. .168 pound.

TO REDUCE A FRACTION TO A DECIMAL.**158.** EXAMPLE.— $\frac{3}{4}$ equals what decimal?SOLUTION.—
$$\begin{array}{r} 4 \overline{) 3.00} \\ \underline{.75} \end{array}, \text{ or } \frac{3}{4} = .75. \quad \text{Ans.}$$
EXAMPLE.—What decimal is equal to $\frac{7}{8}$?SOLUTION.—
$$\begin{array}{r} 8 \overline{) 7.000} .875 \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array} \quad \text{or } \frac{7}{8} = .875. \quad \text{Ans.}$$

159. Rule 17.—*Annex ciphers to the numerator and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed.*

EXAMPLES FOR PRACTICE.**160.** Reduce the following common fractions to decimals:

(a) $\frac{15}{32}$.	Ans. {	(a) .46875.
(b) $\frac{7}{8}$.		(b) .875.
(c) $\frac{21}{32}$.		(c) .65625.
(d) $\frac{51}{64}$.		(d) .796875.
(e) $\frac{4}{25}$.		(e) .16.
(f) $\frac{1}{8}$.		(f) .125.
(g) $\frac{10}{200}$.		(g) .05.
(h) $\frac{4}{1000}$.		(h) .004.

161. To reduce inches to decimal parts of a foot :

EXAMPLE.—What decimal part of a foot is 9 inches?

SOLUTION.—Since there are 12 inches in one foot, 1 inch is $\frac{1}{12}$ of a foot, and 9 inches is $9 \times \frac{1}{12}$ or $\frac{3}{4}$ of a foot. This reduced to a decimal by the above rule, shows what decimal part of a foot 9 inches is.
$$\begin{array}{r} 12 \overline{) 9.00} .75 \text{ of a foot. } \text{Ans.} \\ \underline{84} \\ 60 \\ \underline{60} \\ 0 \end{array}$$

162. Rule 18.—(a) *To reduce inches to decimal parts of a foot, divide the number of inches by 12.*

(b) *Should the resulting decimal be an unending one and it is desired to terminate the division at some point, say the fourth decimal place, carry the division one place farther, and if the fifth figure is 5 or greater, increase the fourth figure by one and omit the sign +.*

EXAMPLES FOR PRACTICE.

163. Reduce to the decimal part of a foot:

(a) 3 in.	Ans. {	(a) .25.
(b) $4\frac{1}{2}$ in.		(b) .375.
(c) 5 in.		(c) .4167.
(d) $6\frac{5}{8}$ in.		(d) .5521.
(e) 11 in.		(e) .9167.

1. The lengths of belting required to connect three countershafts with the main line shaft were found with a tape measure to be 27 ft. 4 in., 23 ft. 8 in., and 38 ft. 6 in. How many feet of belting were necessary? Ans. 89.5 ft.

2. The stroke of an engine is 14 inches. What is the length of the crank measured from the center of shaft to center of crank-pin in feet? Ans. .5833+ foot.

3. A steam pipe fitted with an expansion joint was found to expand 1.668 inches when steam was admitted to it. How much was its expansion in decimal parts of a foot? Ans. .139 foot.

TO REDUCE A DECIMAL TO A FRACTION.

164. EXAMPLE.—Reduce .125 to a fraction.

SOLUTION.— $.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$. Ans.

EXAMPLE.—Reduce .875 to a fraction.

SOLUTION.— $.875 = \frac{875}{1000} = \frac{7}{8}$. Ans.

165. Rule 19.—*Under the figures of the decimal, place the digit 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLES FOR PRACTICE.

166. Reduce the following to common fractions:

(a) .125.	Ans. {	(a) $\frac{1}{8}$.
(b) .625.		(b) $\frac{5}{8}$.
(c) .3125.		(c) $\frac{5}{16}$.
(d) .04.		(d) $\frac{1}{25}$.
(e) .06.		(e) $\frac{3}{50}$.
(f) .75.		(f) $\frac{3}{4}$.
(g) .15625.		(g) $\frac{5}{32}$.
(h) .875.		(h) $\frac{7}{8}$.

**TO EXPRESS A DECIMAL APPROXIMATELY
AS A FRACTION HAVING A GIVEN
DENOMINATOR.**

167. EXAMPLE.—Express .5827 in 64ths.SOLUTION.— $.5827 \times \frac{64}{64} = \frac{37.2928}{64}$, say $\frac{37}{64}$.Hence, $.5827 = \frac{37}{64}$, nearly. Ans.

EXAMPLE.—Express .3917 in 12ths.

SOLUTION.— $.3917 \times \frac{12}{12} = \frac{4.7004}{12}$, say $\frac{5}{12}$.Hence, $.3917 = \frac{5}{12}$, nearly. Ans.

168. Rule 20.—*Reduce 1 to a fraction having the given denominator. Multiply the given decimal by the fraction so obtained, and the result will be the fraction required.*

EXAMPLES FOR PRACTICE.

169. Express

(a) .625 in 8ths.	Ans. {	(a) $\frac{5}{8}$.
(b) .3125 in 16ths.		(b) $\frac{5}{16}$.
(c) .15625 in 32ds.		(c) $\frac{5}{32}$.
(d) .77 in 64ths.		(d) $\frac{39}{64}$.
(e) .81 in 48ths.		(e) $\frac{27}{32}$.
(f) .923 in 96ths.		(f) $\frac{89}{96}$.

170. The sign for dollars is $\$$. It is read dollars. $\$25$ is read 25 dollars.

Since there are 100 cents in a dollar, one cent is 1-one-hundredth of a dollar; the first two figures of a decimal part of a dollar represent *cents*. Since a mill is $\frac{1}{10}$ of a cent, or $\frac{1}{1000}$ of a dollar, the third figure represents mills.

Thus, $\$25.16$ is read twenty-five dollars and sixteen cents; $\$25.168$ is read twenty-five dollars, sixteen cents and eight mills.

171. The **vinculum** —, **parenthesis** (), **bracket** [], and **brace** { } are called **symbols of aggregation**, and are used to include numbers which are to be considered together; thus, $13 \times 8 - 3$, or $13 \times (8 - 3)$, shows that 3 is to be taken from 8 before multiplying by 13.

$$13 \times (8 - 3) = 13 \times 5 = 65. \quad \text{Ans.}$$

$$13 \times 8 - 3 = 13 \times 5 = 65. \quad \text{Ans.}$$

When the vinculum or parenthesis is not used, we have

$$13 \times 8 - 3 = 104 - 3 = 101. \quad \text{Ans.}$$

172. In any series of numbers connected by the signs +, —, $\times$, and $\div$, the operations indicated by the signs must be performed in order from left to right, *except* that no addition or subtraction may be performed if a sign of multiplication or division *follows* the number on the *right* of a sign of addition or subtraction, until the indicated multiplication or division has been performed. In all cases the sign of multiplication takes the precedence, the reason being that when two or more numbers or expressions are connected by the sign of multiplication, the numbers thus connected are regarded as factors of the product indicated, and not as separate numbers.

EXAMPLE.—What is the value of $4 \times 24 - 8 + 17$?

SOLUTION.—Performing the operations in order from left to right, $4 \times 24 = 96$; $96 - 8 = 88$; $88 + 17 = 105$. Ans.

173. EXAMPLE.—What is the value of the following expression : $1,296 \div 12 + 160 - 22 \times 3\frac{1}{2} = ?$

SOLUTION.— $1,296 \div 12 = 108$; $108 + 160 = 268$; here we cannot subtract 22 from 268 because the sign of multiplication *follows* 22; hence, multiplying 22 by $3\frac{1}{2}$, we get 77, and $268 - 77 = 191$. Ans.

Had the above expression been written $1,296 \div 12 + 160 - 22 \times 3\frac{1}{2} \div 7 + 25$, it would have been necessary to have divided $22 \times 3\frac{1}{2}$ by 7 before subtracting, and the final result would have been $22 \times 3\frac{1}{2} = 77$; $77 \div 7 = 11$; $268 - 11 = 257$; $257 + 25 = 282$. Ans. In other words, it is necessary to perform *all* of the multiplication or division included between the signs $+$ and $-$, or $-$ and $+$, before adding or subtracting. Also, had the expression been written $1,296 \div 12 + 160 - 24\frac{1}{2} \div 7 \times 3\frac{1}{2} + 25$, it would have been necessary to have multiplied $3\frac{1}{2}$ by 7 before dividing $24\frac{1}{2}$, since the sign of multiplication takes the precedence, and the final result would have been $3\frac{1}{2} \times 7 = 24\frac{1}{2}$; $24\frac{1}{2} \div 24\frac{1}{2} = 1$; $268 - 1 = 267$; $267 + 25 = 292$. Ans.

It likewise follows that if a succession of multiplication and division signs occurs, the indicated operations must not be performed in order, from left to right—the multiplication must be performed first. Thus, $24 \times 3 \div 4 \times 2 \div 9 \times 5 = \frac{1}{3}$. Ans. In order to obtain the same result that would be obtained by performing the indicated operations in order, from left to right, symbols of aggregation must be used. Thus, by using two vinculum, the last expression becomes $24 \times \overline{3 \div 4} \times \overline{2 \div 9} \times 5 = 20$, the same result that would be obtained by performing the indicated operations in order, from left to right.

EXAMPLES FOR PRACTICE.

174. Find the values of the following expressions :

- | | | |
|---|--------|----------|
| (a) $(8 + 5 - 1) + 4$. | Ans. { | (a) 3. |
| (b) $5 \times 24 - 32$. | | (b) 88. |
| (c) $5 \times 24 + 15$. | | (c) 8. |
| (d) $144 - 5 \times 24$. | | (d) 24. |
| (e) $(1,691 - 540 + 559) + 3 \times 57$. | | (e) 10. |
| (f) $2,080 + 120 - 80 \times 4 - 1,670$. | | (f) 210. |
| (g) $\frac{90 + 60}{90 + 60} \div 25 \times 5 - 29$. | | (g) 1. |
| (h) $\frac{90 + 60}{90 + 60} + 25 \times 5$. | | (h) 1.2. |

ARITHMETIC.

(CONTINUED.)

PERCENTAGE.

175. Percentage is the process of calculating by *hundredths*.

176. The *term per cent.* is an abbreviation of the Latin words *per centum*, which mean *by the hundred*. A certain per cent. of a number is the number of hundreds of that number which is indicated by the number of units in the per cent. Thus, 6 per cent. of 125 is $125 \times \frac{6}{100} = 1.5$; 25 per cent. of 80 is $80 \times \frac{25}{100} = 20$; 43 per cent. of 432 pounds is $432 \times \frac{43}{100} = 185.76$ pounds.

177. The *sign* of per cent. is %, and is read *per cent.* Thus, 6% is read *six per cent.*; $12\frac{1}{2}\%$ is read *twelve and one-half per cent.*, etc.

When expressing the per cent. of a number to use in calculations it is customary to express it decimally instead of fractionally. Thus, instead of expressing 6%, 25%, and 43% as $\frac{6}{100}$, $\frac{25}{100}$, and $\frac{43}{100}$, it is usual to express them as .06, .25, and .43.

The following table will show how any per cent. can be expressed either as a decimal or as a fraction:

TABLE 3.

Per Cent.	Decimal.	Fraction.	Per Cent.	Decimal.	Fraction.
1%.....	.01	$\frac{1}{100}$	150 %....	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%.....	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500 %....	5.00	$\frac{500}{100}$ or 5
5%.....	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$\frac{1}{4}\%$	.0025	$\frac{1}{400}$ or $\frac{1}{400}$
10%.....	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$\frac{1}{2}\%$	.005	$\frac{1}{200}$ or $\frac{1}{200}$
25%.....	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$1\frac{1}{2}\%$	.015	$\frac{15}{1000}$ or $\frac{3}{200}$
50%.....	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$8\frac{1}{3}\%$	.08 $\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%.....	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$12\frac{1}{2}\%$	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%.....	1.00	$\frac{100}{100}$ or 1	$16\frac{2}{3}\%$	.16 $\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%.....	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	$62\frac{1}{2}\%$	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

178. The names of the different elements used in percentage are : the *base*, the *rate per cent.*, the *percentage*, the *amount*, and the *difference*.

179. The **base** is the number on which the per cent. is computed.

180. The **rate** is the number of hundredths of the base to be taken.

181. The **percentage** is the part, or number of *hundredths*, of the base indicated by the rate ; or the percentage is the result obtained by multiplying the base by the rate.

Thus, when it is stated that 7% of \$25 is \$1.75, \$25 is the base, 7% is the rate, and \$1.75 is the percentage.

182. The **amount** is the sum of the base and percentage.

183. The **difference** is the remainder obtained by subtracting the percentage from the base.

Thus, if a man has \$180, and he earns 6% more, he will have altogether $\$180 + \$180 \times .06$, or $\$180 + \$10.80 = \$190.80$. Here \$180 is the base, 6% the rate, \$10.80 the percentage, and \$190.80 the *amount*.

Again, if an engine of 125 horsepower uses 16% of it in overcoming friction and other resistances, the amount left for performing useful work is $125 - 125 \times .16 = 125 - 20 = 105$ horsepower. Here 125 is the base, 16% the rate, 20 the percentage, and 105 the *difference*.

184. From the foregoing it is evident that, to find the percentage, the base must be multiplied by the rate. Hence the following

Rule 21.—*To find the percentage, multiply the base by the rate, expressed decimally.*

EXAMPLE.—Out of a lot of 300 boiler tubes 76% were used in a boiler. How many tubes were used ?

SOLUTION.—76%, the rate, expressed decimally, is .76 ; the base is 300 ; hence, the number of tubes used, or the percentage, is by the above rule $300 \times .76 = 228$ tubes. Ans.

Expressing the rule as a formula, we have

$$\text{percentage} = \text{base} \times \text{rate}.$$

185. When the percentage and rate are given, the base may be found by dividing the percentage by the rate. For, suppose that 12 is 6%, or $\frac{6}{100}$, of some number; then, 1%, or $\frac{1}{100}$, of the number is $12 \div 6$ or 2. Consequently, if $2 = 1\%$, or $\frac{1}{100}$, 100%, or $\frac{100}{100} = 2 \times 100 = 200$. But, since the same result may be arrived at by dividing 12 by .06, for $12 \div .06 = 200$ it follows that

Rule 22.—*When the percentage and rate are given, to find the base, divide the percentage by the rate, expressed decimally.*

Formula, $\text{base} = \text{percentage} \div \text{rate}$.

EXAMPLE.—76% of a lot of boiler tubes are used in the construction of a boiler. If the number of tubes used were 228, how many tubes were in the lot?

SOLUTION.—Here 228 is the percentage, and 76%, or .76, is the rate; hence, applying the rule,

$$228 \div .76 = 300 \text{ tubes. Ans.}$$

186. When the base and percentage are given to find the rate, the rate may be found, expressed decimally, by dividing the percentage by the base. For suppose that it is desired to find what per cent. 12 is of 200. 1% of 200 is $200 \times .01 = 2$. Now, if 1% is 2, 12 is evidently as many per cent. as the number of times that 2 is contained in 12, or $12 \div 2 = 6\%$. But the same result may be obtained by dividing 12, the percentage, by 200, the base, since $12 \div 200 = .06 = 6\%$. Hence,

Rule 23.—*When the percentage and base are given, to find the rate, divide the percentage by the base, and the result will be the rate, expressed decimally.*

Formula, $\text{rate} = \text{percentage} \div \text{base}$.

EXAMPLE.—Out of a lot of 300 boiler tubes, 228 were used. What per cent. of the total number was used?

SOLUTION.—Here 300 is the base and 228 the percentage; hence, applying rule,

$$\text{rate} = 228 \div 300 = .76 = 76\%. \text{ Ans.}$$

EXAMPLE.—What per cent. of 875 is 25?

SOLUTION.—Here 875 is the base, and 25 is the percentage; hence, applying rule,

$$25 \div 875 = .02\frac{2}{7} = 2\frac{2}{7}\%. \text{ Ans.}$$

PROOF.— $875 \times .02\frac{2}{7} = 25$.

EXAMPLES FOR PRACTICE.

187. What per cent. of

- (a) 360 is 90 ?
 (b) 900 is 360 ?
 (c) 125 is 25 ?
 (d) 150 is 750 ?
 (e) 280 is 112 ?
 (f) 400 is 200 ?
 (g) 47 is 94 ?
 (h) 500 is 250 ?

Ans. $\left\{ \begin{array}{l} (a) 25\%. \\ (b) 40\%. \\ (c) 20\%. \\ (d) 500\%. \\ (e) 40\%. \\ (f) 50\%. \\ (g) 200\%. \\ (h) 50\%. \end{array} \right.$

188. The amount may be found when the base and rate are given, by multiplying the base by 1 plus the rate, expressed decimally. For suppose that it is desired to find the amount when 200 is the base and 6% is the rate. The percentage is $200 \times .06 = 12$, and, according to definition, Art. 182, the amount is $200 + 12 = 212$. But the same result may be obtained by multiplying 200 by $1 + .06$, or 1.06, since $200 \times 1.06 = 212$. Hence,

Rule 24.—*When the base and rate are given, to find the amount, multiply the base by 1 plus the rate, expressed decimally.*

Formula, $\text{amount} = \text{base} \times (1 + \text{rate})$.

EXAMPLE.—If a man earned \$725 in a year, and the next year 10% more, how much did he earn the second year ?

SOLUTION.—Here 725 is the base and 10% is the rate, and the amount is required. Hence, applying the rule,

$$725 \times 1.10 = \$797.50. \quad \text{Ans.}$$

189. When the base and rate are given, the difference may be found by multiplying the base by 1 minus the rate, expressed decimally. For suppose that it is desired to find the difference when the base is 200 and the rate is 6%. The percentage is $200 \times .06 = 12$; and, according to definition, Art. 183, the difference $= 200 - 12 = 188$. But the same result may be obtained by multiplying 200 by $1 - .06$, or .94, since $200 \times .94 = 188$. Hence,

Rule 25.—*When the base and rate are given, to find the difference, multiply the base by 1 minus the rate, expressed decimally.*

Formula, $\text{difference} = \text{base} \times (1 - \text{rate})$.

EXAMPLE.—Out of a lot of 300 boiler tubes all but 24% were used in one boiler; how many tubes were used?

SOLUTION.—Here 300 is the base, 24% is the rate, and it is desired to find the difference. Hence, applying the rule.

$$300 \times (1 - .24) = 228 \text{ tubes. Ans.}$$

190. When the amount and rate are given, the base may be found by dividing the amount by 1 plus the rate. For suppose that it is known that 212 equals some number increased by 6% of itself. Then, it is evident that 212 equals 106% of the number (base) that it is desired to find. Consequently, if $212 = 106\%$, $1\% = \frac{212}{106} = 2$, and $100\% = 2 \times 100 = 200 = \text{the base}$. But the same result may be obtained by dividing 212 by $1 + .06$ or 1.06, since $212 \div 1.06 = 200$. Hence,

Rule 26.—*When the amount and rate are given, to find the base, divide the amount by 1 plus the rate, expressed decimally.*

$$\text{Formula, base} = \text{amount} \div (1 + \text{rate}).$$

EXAMPLE.—The theoretical discharge of a certain pump when running at a piston speed of 100 feet per minute is 278,910 gallons per day of 10 hours. Owing to leakage and other defects this value is 25% greater than the actual discharge. What is the actual discharge?

SOLUTION.—Here 278,910 equals the actual discharge (base) increased by 25% of itself. Consequently, 278,910 is the amount, 25% is the rate, and applying rule,

$$\text{actual discharge} = 278,910 \div 1.25 = 223,128 \text{ gallons. Ans.}$$

191. When the difference and rate are given, the base may be found by dividing the difference by 1 minus the rate. For suppose that 188 equals some number less 6% of itself. Then, 188 evidently equals $100 - 6 = 94\%$ of some number. Consequently, if $188 = 94\%$, $1\% = 188 \div 94 = 2$, and $100\% = 2 \times 100 = 200$. But the same result may be obtained by dividing 188 by $1 - .06$, or .94, since $188 \div .94 = 200$. Hence,

Rule 27.—*When the difference and rate are given, to find the base, divide the difference by 1 minus the rate, expressed decimally.*

$$\text{Formula, base} = \text{difference} \div (1 - \text{rate}).$$

EXAMPLE.—From a lot of boiler tubes 76% were used in the construction of a boiler. If there were 72 tubes unused, how many tubes were in the lot?

SOLUTION.—Here 72 is the difference and 76% is the rate. Applying rule,

$$72 \div (1 - .76) = 300 \text{ tubes. Ans.}$$

EXAMPLE.—The theoretical number of foot-pounds of work per minute required to operate a boiler feed-pump is 127,344. If 30% of the total number actually required be allowed for friction, leakage, etc., how many foot-pounds are actually required to work the pump?

SOLUTION.—Here the number actually required is the base; hence, 127,344 is the difference, and 30% is the rate. Applying the rule,

$$127,344 \div (1 - .30) = 181,920 \text{ foot-pounds. Ans.}$$

192. EXAMPLE.—A certain chimney gave a draft of 2.76 inches of water. By increasing the height 20 feet the draft was increased to 3 inches of water. What was the gain per cent.?

SOLUTION.—Here it is evident that 3 inches is the amount and that 2.76 inches is the base. Consequently, $3 - 2.76 = .24$ inch is the percentage, and it is required to find the rate. Hence, applying rule **23**,

$$\text{gain per cent.} = .24 \div 2.76 = .087 = 8.7\%. \text{ Ans.}$$

193. EXAMPLE.—A certain chimney gave a draft of 3 inches of water. After an economizer had been put in the draft was reduced to 1.2 inches of water. What was the loss per cent.?

SOLUTION.—Here it is evident 1.2 inches is the difference (since it equals 3 inches diminished by a certain per cent. loss of itself) and 3 inches is the base. Consequently, $3 - 1.2 = 1.8$ inches is the percentage. Hence, applying rule **23**,

$$\text{loss per cent.} = 1.8 \div 3 = .60 = 60\%. \text{ Ans.}$$

194. To find the gain or loss per cent.:

Rule 28.—*Find the difference between the initial and final values; divide this difference by the initial value.*

EXAMPLE.—If a man buys a steam engine for \$1,860 and some time afterwards purchases a condenser for 25% of the cost of the engine, does he gain or lose, and how much per cent., if he sells both engine and condenser for \$2,100?

SOLUTION.—The cost of the condenser was $\$1,860 \times .25 = \$465$; consequently, the total initial value, or cost, was $\$1,860 + \$465 = \$2,325$. Since he sold them for \$2,100, he lost $\$2,325 - \$2,100 = \$225$. Hence, applying rule,

$$225 \div 2,325 = .0968 = 9.68\% \text{ loss. Ans.}$$

EXAMPLES FOR PRACTICE.

195. Solve the following:

(a) What is $12\frac{1}{2}\%$ of \$900?	Ans.	(a) \$112.50.
(b) " " $\frac{1}{4}\%$ " 627?		(b) 5.016.
(c) " " $33\frac{1}{3}\%$ " 54?		(c) 18.
(d) 101 is $68\frac{1}{2}\%$ of what number?		(d) 146 $\frac{1}{3}$.
(e) 784 " $83\frac{1}{4}\%$ " " "		(e) 940.8.
(f) What % of 960 is 160?		(f) $16\frac{2}{3}\%$.
(g) " " " \$3,606 is \$450 $\frac{1}{4}$?		(g) $12\frac{1}{4}\%$.
(h) " " " 280 is 112?		(h) 40%.

1. A steam plant consumed an average of 3,640 pounds of coal per day. The engineer made certain alterations which resulted in a saving of 250 pounds per day. What was the per cent. of coal saved?

Ans. 7%, nearly.

2. If the speed of an engine running at 126 revolutions per minute should be increased $6\frac{1}{4}\%$, how many revolutions per minute would it then make?

Ans. 134.19 revolutions.

3. The list price of an engine was \$1,400; of a boiler, \$1,150, and of the necessary fittings for the two, \$340. If 25% discount was allowed on the engine, 22% on the boiler, and $12\frac{1}{2}\%$ on the fittings, what was the actual cost of the plant?

Ans. \$2,244.50.

4. If I lend a man \$1,100, and this is $18\frac{1}{2}\%$ of the amount that I have on interest, how much money have I on interest?

Ans. \$5,945.95.

5. A test showed that an engine developed 190.4 horsepower, 15% of which was consumed in friction. How much power was available for use?

Ans. 161.84 H. P.

6. By adding a condenser to a steam engine, the power was increased 14%, and the consumption of coal per horsepower per hour was decreased 20%. If the engine could originally develop 50 horsepower and required $3\frac{1}{4}$ pounds of coal per horsepower per hour, what would be the total weight of coal used in an hour, with the condenser, assuming the engine to run full power?

Ans. 159.6 pounds.

DENOMINATE NUMBERS.

196. A **denominate number** is a concrete number, and may be either simple or compound, as 8 quarts, 5 feet, ten inches, etc.

197. A **simple denominate number** consists of units of but one denomination, as 16 cents, 10 hours, 5 dollars, etc.

198. A **compound denominate number** consists of units of two or more denominations of a similar kind, as 3 yards, 2 feet, 1 inch ; 3 pounds, 5 ounces.

199. In **whole numbers** and in **decimals**, the *law* of increase and decrease is on the scale of 10, but in **compound** or **denominate numbers** the scale varies.

MEASURES.

200. A **measure** is a *standard unit* established by *law* or *custom*, by which *quantity* of any kind is measured. The *standard unit* of **dry measure** is the Winchester bushel; of **weight**, the pound; of **liquid measure**, the gallon, etc.

201. Measures are of six kinds:

- | | |
|---------------|--------------------|
| 1. Extension. | 4. Time. |
| 2. Weight. | 5. Angles. |
| 3. Capacity. | 6. Money or value. |

MEASURES OF EXTENSION.

202. **Measures of extension** are used in measuring lengths, distances, surfaces, and solids.

LINEAR MEASURE.

TABLE 4.

	Abbreviation.				
12 inches (in.)	= 1 foot . . ft.	in.	ft.	yd.	rd.
3 feet	= 1 yard . . yd.	36	=	3	
5½ yards	= 1 rod . . rd.	198	=	16½	= 5½
40 rods	= 1 furlong . fur.	7,920	=	660	= 220 = 40
8 furlongs	= 1 mile . . mi.	63,360	=	5,280	= 1,760 = 320 = 8

SQUARE MEASURE.

TABLE 5.

144 square inches (sq. in.)	. . . =	1 square foot	sq. ft.
9 square feet	=	1 square yard	sq. yd.
30½ square yards	=	1 square rod	sq. rd.
160 square rods	=	1 acre	A.
640 acres	=	1 square mile	sq. mi.
sq. mi.	A.	sq. rd.	sq. yd.
1	= 640 = 102,400 = 3,097,600 = 27,878,400 = 4,014,489,600	sq. ft.	sq. in.

CUBIC MEASURE.

TABLE 6.

1728 cubic inches (cu. in.) . . .	=	1 cubic foot	cu. ft.
27 cubic feet	=	1 cubic yard	cu. yd.
128 cubic feet	=	1 cord	cd.
24½ cubic feet	=	1 perch	P.
cu. yd. cu. ft. cu. in.			
1 = 27 = 46,656			

MEASURES OF WEIGHT.

AVOIRDUPOIS WEIGHT.

TABLE 7.

16 ounces (oz.)	=	1 pound	lb.
100 pounds	=	1 hundred-weight	cwt.
20 cwt., or 2,000 lb.	=	1 ton	T.
T. cwt. lb. oz.			
1 = 20 = 2,000 = 32,000			

203. The ounce is divided into halves, quarters, etc. Avoirdupois weight is used for weighing coarse and heavy articles.

LONG TON TABLE.

TABLE 8.

16 ounces	=	1 pound	lb.
112 pounds	=	1 hundred-weight	cwt.
20 cwt., or 2,240 lb.	=	1 ton	T.

In all the calculations throughout this and the succeeding volumes, 2,000 pounds will be considered one ton, unless the long ton (2,240 pounds) is especially mentioned.

TROY WEIGHT.

TABLE 9.

24 grains (gr.)	=	1 pennyweight	pwt.
20 pennyweight	=	1 ounce	oz.
12 ounces	=	1 pound	lb.
lb. oz. pwt. gr.			
1 = 12 = 240 = 5,760			

204. Troy weight is used in weighing gold and silver ware, jewels, etc. It is used by jewelers.

MEASURES OF CAPACITY.**LIQUID MEASURE.**

TABLE 10.

4 gills (gi.)	=	1 pint	pt.							
2 pints	=	1 quart.	qt.							
4 quarts	=	1 gallon	gal.							
31½ gallons	=	1 barrel	bbl.							
63 gallons	=	1 hogshead	hhd.							
hhd.	bbl.	gal.	qt.	pt.	gi.					
1	=	2	=	63	=	252	=	504	=	2,016

DRY MEASURE.

TABLE 11.

2 pints (pt.)	=	1 quart	qt.
8 quarts	=	1 peck	pk.
4 pecks	=	1 bushel	bu.
	bu.	pk.	qt.
	1	= 4	= 32 = 64

MEASURE OF TIME.

TABLE 12.

60 seconds (sec.)	=	1 minute	min.
60 minutes	=	1 hour	hr.
24 hours	=	1 day	da.
7 days	=	1 week	wk.
365 days }	=	1 common year	yr.
12 months }			
366 days	=	1 leap year.	
100 years	=	1 century.	

NOTE.—It is customary to consider one month as 30 days.

MEASURE OF ANGLES OR ARCS.

TABLE 13.

60 seconds (")	=	1 minute	'
60 minutes	=	1 degree	°
90 degrees	=	1 right angle or quadrant	┘
360 degrees	=	1 circle	cir.
cir.			
1	=	360°	= 21,600' = 1,296,000''

MEASURE OF MONEY.
UNITED STATES MONEY.

TABLE 14.

10 mills (m.)	=	1 cent	ct.
10 cents	=	1 dime	d.
10 dimes	=	1 dollar	\$
10 dollars	=	1 eagle	E
E.	\$	d.	ct.
1	=	10	=
		100	=
		1,000	=
		10,000	

MISCELLANEOUS TABLE.

TABLE 15.

12 things are 1 dozen.	1 meter is nearly 39.37 inches.
12 dozen are 1 gross.	1 hand is 4 inches.
12 gross are 1 great gross.	1 palm is 3 inches.
2 things are 1 pair.	1 span is 9 inches.
20 things are 1 score.	24 sheets are 1 quire.
1 league is 3 miles.	20 quires, or 480 sheets, are 1 ream.
1 fathom is 6 feet.	1 bushel contains 2,150.4 cubic in.
1 U. S. standard gallon (also called a wine gallon) contains 231 cubic in.	
1 U. S. standard gallon of water weighs 8.355 pounds, nearly.	
1 cubic foot of water contains 7.481 U. S. standard gallons, nearly.	
1 British imperial gallon weighs 10 pounds.	

It will be of great advantage to the student to carefully memorize all of the above tables.

REDUCTION OF DENOMINATE NUMBERS.

205. Reduction of denominate numbers is the process of changing their denomination without changing their value. They may be changed from a higher to a lower denomination or from a lower to a higher—either is reduction. As,

$$2 \text{ hours} = 120 \text{ minutes.}$$

$$32 \text{ ounces} = 2 \text{ pounds.}$$

206. Principle.—Denominate numbers are changed to *lower* denominations by *multiplying*, and to *higher* denominations by *dividing*.

To reduce denominate numbers to lower denominations:

207. EXAMPLE.—Reduce 5 yd. 2 ft. 7 in. to inches.

SOLUTION.—

yd.	ft.	in.
5	2	7
3		
15 ft.		
2 ft.		
17 ft.		
12		
34		
17		
204 in.		
	7 in.	
211 inches.		Ans.

EXPLANATION.—Since there are 3 feet in 1 yard, in 5 yards there are 5×3 , or 15 feet, and 15 feet plus 2 feet = 17 feet. There are 12 inches in a foot; therefore, $12 \times 17 = 204$ inches, and 204 inches plus 7 inches = 211 inches = number of inches in 5 yards 2 feet and 7 inches. Ans.

208. EXAMPLE.—Reduce 6 hours to seconds.

SOLUTION.—

6	hours.
60	
360	minutes.
60	
21600	seconds. Ans.

EXPLANATION.—As there are 60 minutes in one hour, in six hours there are 6×60 , or 360 minutes; as there are no minutes to add, we multiply 360 minutes by 60, to get the number of seconds.

209. In order to avoid mistakes, if any denomination be omitted, represent it by a cipher. Thus, before reducing 3 rods 6 inches to inches, insert a cipher for yards and a cipher for feet; as,

rd.	yd.	ft.	in.
3	0	0	6

210. Rule.—Multiply the number representing the highest denomination by the number of units in the next lower

required to make one of the higher denomination, and to the product add the number of given units of that lower denomination. Proceed in this manner until the number is reduced to the required denomination.

EXAMPLES FOR PRACTICE.

211. Reduce

(a) 4 rd. 2 yd. 2 ft. to ft.	Ans. {	(a) 74 ft.
(b) 4 bu. 3 pk. 2 qt. to qt.		(b) 154 qt.
(c) 13 rd. 5 yd. 2 ft. to ft.		(c) 231.5 ft.
(d) 5 mi. 100 rd. 10 ft. to ft.		(d) 28,060 ft.
(e) 8 lb. 4 oz. 6 pwt. to gr.		(e) 48,144 gr.
(f) 52 hhd. 24 gal. 1 pt. to pt.		(f) 26,401 pt.
(g) 5 cir. 16° 20' to minutes.		(g) 108,980'.
(h) 14 bu. to qt.		(h) 448 qt.

To reduce lower to higher denominations:

212. EXAMPLE.—Reduce 211 in. to higher denominations.

$$\begin{array}{r}
 \text{SOLUTION.—} \quad 12 \overline{) 211 \text{ in.}} \\
 \underline{3 \overline{) 17 \text{ ft.} + 7 \text{ in.}}} \\
 5 \text{ yd.} + 2 \text{ ft.} \quad \text{Ans.}
 \end{array}$$

EXPLANATION.—There are 12 inches in 1 foot; therefore, 211 divided by 12 = 17 feet and 7 inches over. There are 3 ft. in 1 yd.; therefore, 17 ft. divided by 3 = 5 yd. and 2 ft. over. The last quotient and the two remainders constitute the answer, 5 yd. 2 ft. 7 in.

213. EXAMPLE.—Reduce 14,135 gi. to higher denominations.

$$\begin{array}{r}
 \text{SOLUTION.—} \quad 4 \overline{) 14135} \\
 2 \overline{) 8533 \text{ pt.} 3 \text{ gi.}} \\
 4 \overline{) 1766 \text{ qt.} 1 \text{ pt.}} \\
 \underline{441 \text{ gal.} 2 \text{ qt.}} \\
 315 \overline{) 441.0} (14 \text{ bbl.} \\
 \underline{315} \\
 1260 \\
 \underline{1260}
 \end{array}$$

EXPLANATION.—There are 4 gi. in 1 pt., and in 14,135 gi. there are as many pints as 4 is contained in 14,135, or 3,533

pt. and 3 gi. remaining. There are 2 pt. in 1 qt., and in 3,533 pt. there are 1,766 qt. and 1 pt. remaining. There are 4 qt. in 1 gal., and in 1,766 qt. there are 441 gal. and 2 qt. remaining. There are $31\frac{1}{2}$ gal. in 1 bbl., and in 441 gal. there are 14 bbl.

The last quotient and the three remainders constitute the answer, 14 bbl. 2 qt. 1 pt. 3 gi.

214. Rule 30.—*Divide the number representing the denomination given, by the number of units of this denomination required to make one unit of the next higher denomination. The remainder will be of the same denomination, but the quotient will be of the next higher. Divide this quotient by the number of units of its denomination required to make one unit of the next higher. Continue until the highest denomination is reached, or until there is not enough of a denomination left to make one of the next higher. The last quotient and the remainders constitute the required result.*

EXAMPLES FOR PRACTICE.

215. Reduce to units of higher denominations:

- (a) 7,460 sq. in.; (b) 7,580 sq. yd.; (c) 148,760 cu. in.; (d) 17,651";
(e) 8,000 gi.; (f) 36,450 lb.

Ans. $\left\{ \begin{array}{l} (a) \text{ 5 sq. yd. 6 sq. ft. 116 sq. in.} \\ (b) \text{ 1 A. 90 sq. rd. 17 sq. yd. 4 sq. ft. 72 sq. in.} \\ (c) \text{ 3 cu. yd. 5 cu. ft. 152 cu. in.} \\ (d) \text{ 4}^\circ \text{ 54' 11".} \\ (e) \text{ 3 hhd. 61 gal.} \\ (f) \text{ 18 T. 4 cwt. 50 lb.} \end{array} \right.$

ADDITION OF DENOMINATE NUMBERS.

216. EXAMPLE.—Find the sum of 3 cwt. 46 lb. 12 oz.; 8 cwt. 12 lb. 13 oz.; 12 cwt. 50 lb. 13 oz.; 27 lb. 4 oz.

SOLUTION.—	T.	cwt.	lb.	oz.
	0	3	46	12
	0	8	12	13
	0	12	50	13
	0	0	27	4
	1	4	37	10
				Ans.

EXPLANATION.—Begin to add at the right-hand column : $4 + 13 + 13 + 12 = 42$ ounces ; as 16 ounces make 1 pound, $42 \text{ ounces} \div 16 = 2$ and a remainder of 10 ounces, or 2 pounds and 10 ounces. Place 10 ounces under ounce column and add 2 pounds to the next or pound column. Then, $2 + 27 + 50 + 12 + 46 = 137$ pounds ; as 100 pounds make a hundred-weight, $137 \div 100 = 1$ hundred-weight and a remainder of 37 pounds. Place the 37 under the pound column, and add 1 hundred-weight to the next or hundred-weight column. Next, $1 + 12 + 8 + 3 = 24$ hundred-weight. 20 hundred-weight make a ton ; therefore $24 \div 20 = 1$ ton and 4 hundred-weight remaining. Hence, the sum is 1 ton 4 hundred-weight 37 pounds 10 ounces. Ans.

217. EXAMPLE.—What is the sum of 2 rd. 3 yd. 2 ft. 5 in. ; 6 rd. 1 ft. 10 in. ; 17 rd. 11 in. ; 4 yd. 1 ft. ?

SOLUTION.—	rd.	yd.	ft.	in.
	2	3	2	5
	6	0	1	10
	17	0	0	11
	0	4	1	0
	26	$3\frac{1}{2}$	0	2
or	26	3	1	8 Ans.

EXPLANATION.—The sum of the numbers in the first column = 26 inches, or 2 feet and 2 inches remaining. The sum of the numbers in the next column plus 2 feet = 6 feet, or 2 yards and 0 feet remaining. The sum of the next column plus 2 yards = 9 yards, or $9 \div 5\frac{1}{2} = 1$ rod and $3\frac{1}{2}$ yards remaining. The sum of the next column plus 1 rod = 26 rods. To avoid fractions in the sum, the $\frac{1}{2}$ yard is reduced to 1 foot and 6 inches, which added to 26 rods 3 yards 0 feet and 2 inches = 26 rods 3 yards 1 foot 8 inches. Ans.

218. EXAMPLE.—What is the sum of 47 ft. and 3 rd. 2 yd. 2 ft. 10 in. ?

SOLUTION.—When 47 ft. is reduced it equals 2 rd. 4 yd. 2 ft., which can be added to 3 rd. 2 yd. 2 ft. 10 in. Thus,

rd.	yd.	ft.	in.	
3	2	2	10	
2	4	2	0	
6	1½	1	10	
or 6	2	0	4	Ans.

219. Rule 31.—Place the numbers so that like denominations are under each other. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the next higher. Place the remainder under the column added, and carry the quotient to the next column. Continue in this manner until the highest denomination given is reached.

EXAMPLES FOR PRACTICE.

220. What is the sum of

(a) 25 lb. 7 oz. 15 pwt. 23 gr.; 17 lb. 16 pwt.; 15 lb. 4 oz. 12 pwt.; 18 lb. 16 gr.; 10 lb. 2 oz. 11 pwt. 16 gr.?

(b) 9 mi. 13 rd. 4 yd. 2 ft.; 16 rd. 5 yd. 1 ft. 5 in.; 16 mi. 2 rd. 3 in.; 14 rd. 1 yd. 9 in.?

(c) 3 cwt. 46 lb. 12 oz.; 12 cwt. 9½ lb.; 2½ cwt. 21¾ lb.?

(d) 10 yr. 8 mo. 5 wk. 3 da.; 42 yr. 6 mo. 7 da.; 7 yr. 5 mo. 18 wk. 4 da.; 17 yr. 17 da.?

(e) 17 tons 11 cwt. 49 lb. 14 oz.; 16 tons 47 lb. 13 oz.; 20 tons 13 cwt. 14 lb. 6 oz.; 11 tons 4 cwt. 16 lb. 12 oz.?

(f) 14 sq. yd. 8 sq. ft. 19 sq. in.; 105 sq. yd. 16 sq. ft. 240 sq. in.; 42 sq. yd. 28 sq. ft. 165 sq. in.?

Ans. { (a) 86 lb. 3 oz. 16 pwt. 7 gr.
 (b) 25 mi. 47 rd. 1 ft. 5 in.
 (c) 18 cwt. 2 lb. 14 oz.
 (d) 78 yr. 1 mo. 3 wk. 3 da.
 (e) 65 tons 9 cwt. 28 lb. 13 oz.
 (f) 167 sq. yd. 136 sq. in.

SUBTRACTION OF DENOMINATE NUMBERS.

221. EXAMPLE.—From 21 rd. 2 yd. 2 ft. 6½ in., take 9 rd. 4 yd. 10½ in.

SOLUTION.—

rd.	yd.	ft.	in.	
21	2	2	6½	
9	4	0	10½	
11	3½	1	8½	Ans.

EXPLANATION.—Since $10\frac{1}{4}$ inches can not be taken from $6\frac{1}{2}$ inches, we must borrow 1 foot, or 12 inches, from the 2 feet in the next column and add it to the $6\frac{1}{2}$. $6\frac{1}{2} + 12 = 18\frac{1}{2}$. $18\frac{1}{2}$ inches $- 10\frac{1}{4}$ inches $= 8\frac{1}{4}$ inches. Then, 0 foot from the 1 remaining foot $= 1$ foot. 4 yards can not be taken from 2 yards; therefore, we borrow 1 rod, or $5\frac{1}{2}$ yards, from 21 rods and add it to 2. $2 + 5\frac{1}{2} = 7\frac{1}{2}$; $7\frac{1}{2} - 4 = 3\frac{1}{2}$ yards. 9 rods from 20 rods $= 11$ rods. Hence, the remainder is 11 rods $3\frac{1}{2}$ yards 1 foot $8\frac{1}{4}$ inches. Ans.

To avoid fractions as much as possible, we reduce the $\frac{1}{4}$ yard to inches, obtaining 18 inches; this added to $8\frac{1}{4}$ inches, gives $26\frac{1}{4}$ inches, which equals 2 feet $2\frac{1}{4}$ inches. Then, 2 feet $+ 1$ foot $= 3$ feet $= 1$ yard, and 3 yards $+ 1$ yard $= 4$ yards. Hence, the above answer becomes 11 rods 4 yards 0 feet $2\frac{1}{4}$ inches.

222. EXAMPLE.—What is the difference between 3 rd. 2 yd. 2 ft. 10 in. and 47 ft. ?

SOLUTION.—47 ft. $= 2$ rd. 4 yd. 2 ft.

rd.	yd.	ft.	in.
3	2	2	10
2	4	2	0
0	$3\frac{1}{4}$	0	10
or	3	2	4 Ans.

To find (approximately) the interval of time between two dates:

223. EXAMPLE.—How many years, months, days, and hours between 4 o'clock P.M. of June 15, 1868, and 10 o'clock A.M., September 28, 1891 ?

SOLUTION.—	yr.	mo.	da.	hr.
	1891	8	28	10
	1868	5	15	16
	23	3	12	18 Ans.

EXPLANATION.—Counting 24 hours in 1 day, 4 o'clock P.M. is the 16th hour from the beginning of the day, or midnight. On September 28, 8 months and 28 days have elapsed, and on June 15, 5 months and 15 days. After placing the earlier date under the later date, subtract as in the previous problems. Count 30 days as 1 month.

224. Rule 32.—Place the smaller quantity under the larger quantity, with like denominations under each other. Beginning at the right, subtract successively the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be borrowed from the minuend of the next higher denomination, reduced, and added to it.

EXAMPLES FOR PRACTICE.

225. From

- (a) 125 lb. 8 oz. 14 pwt. 18 gr. take 96 lb. 9 oz. 10 pwt. 4 gr.
 (b) 126 hhd. 27 gal. take 104 hhd. 14 gal. 1 qt. 1 pt.
 (c) 65 T. 14 cwt. 64 lb. 10 oz. take 16 T. 11 cwt. 14 oz.
 (d) 148 sq. yd. 16 sq. ft. 142 sq. in. take 132 sq. yd. 136 sq. in.
 (e) 100 bu. take 28 bu. 2 pk. 5 qt. 1 pt.
 (f) 14 mi. 34 rd. 16 yd. 13 ft. 11 in. take 3 mi. 27 rd. 11 yd. 4 ft. 10 in.

Ans. $\left\{ \begin{array}{l} (a) \text{ 28 lb. 11 oz. 4 pwt. 14 gr.} \\ (b) \text{ 22 hhd. 12 gal. 2 qt. 1 pt.} \\ (c) \text{ 49 T. 3 cwt. 63 lb. 12 oz.} \\ (d) \text{ 16 sq. yd. 16 sq. ft. 6 sq. in.} \\ (e) \text{ 71 bu. 1 pk. 2 qt. 1 pt.} \\ (f) \text{ 11 mi. 7 rd. 5 yd. 9 ft. 1 in.} \end{array} \right.$

MULTIPLICATION OF DENOMINATE NUMBERS.

226. EXAMPLE.—Multiply 7 lb. 5 oz. 13 pwt. 15 gr. by 12.

SOLUTION.—	lb.	oz.	pwt.	gr.
	7	5	13	15
				12
	89	8	3	12
				Ans.

EXPLANATION.—15 grains $\times$ 12 = 180 grains. $180 \div 24 = 7$ pennyweights and 12 grains remaining. Place the 12 in the grain column and carry the 7 pennyweights to the next. Now, $13 \times 12 + 7 = 163$ pennyweights; $163 \div 20 = 8$ ounces and 3 pennyweights remaining. Then, $5 \times 12 + 8 = 68$ ounces; $68 \div 12 = 5$ pounds and 8 ounces remaining. Then, $7 \times 12 + 5 = 89$ pounds. The entire product is 89 pounds 8 ounces 3 pennyweights 12 grains. Ans.

227. Rule 33.—*Multiply the number representing each denomination by the multiplier, and reduce each product to the next higher denomination, writing the remainders under each denomination, and carrying the quotient to the next, as in addition of denominate numbers.*

228. NOTE.—In multiplication and division of denominate numbers, it is sometimes easier to reduce the number to the lowest denomination given before multiplying or dividing, especially if the multiplier or divisor is a decimal. Thus, in the above example, had the multiplier been 1.2, the easiest way to multiply would have been to reduce the number to grains, then, multiply by 1.2, and reduce the product to higher denominations. For example, 7 lb. 5 oz. 13 pwt. 15 gr. = 43,047 gr. $43,047 \times 1.2 = 51,656.4$ gr. = 8 lb. 11 oz. 12 pwt. 8.4 gr. Also, $43,047 \times 12 = 516,564$ gr. = 89 lb. 8 oz. 3 pwt. 12 gr., as above. The student may use either method.

EXAMPLES FOR PRACTICE.

229. Multiply

(a) 15 cwt. 90 lb. by 5; (b) 12 yr. 10 mo. 4 wk. 3 da. by 14; (c) 11 mi. 145 rd. by 20; (d) 12 gal. 4 pt. by 9; (e) 8 cd. 76 cu. ft. by 15; (f) 4 hhd. 3 gal. 1 qt. 1 pt. by 12.

Ans. $\left\{ \begin{array}{l} (a) \text{ 79 cwt. 50 lb.} \\ (b) \text{ 180 yr. 11 mo. 2 wk.} \\ (c) \text{ 229 mi. 20 rd.} \\ (d) \text{ 112 gal. 2 qt.} \\ (e) \text{ 128 cd. 116 cu. ft.} \\ (f) \text{ 48 hhd. 40 gal. 2 qt.} \end{array} \right.$

DIVISION OF DENOMINATE NUMBERS.

230. EXAMPLE.—Divide 48 lb. 11 oz. 6 pwt. by 8.

SOLUTION.—	lb.	oz.	pwt.	gr.	
8) 48	11	6	0		
	6 lb.	1 oz.	8 pwt.	6 gr.	Ans.

EXPLANATION.—After placing the quantities as above, proceed as follows: 8 is contained in 48 six times without a remainder. 8 is contained in 11 ounces once with 3 ounces remaining. $3 \times 20 = 60$; $60 + 6 = 66$ pennyweights; $66 \text{ pennyweights} \div 8 = 8 \text{ pennyweights}$ and 2 remaining; $2 \times 24 \text{ grains} = 48 \text{ grains}$; $48 \text{ grains} \div 8 = 6 \text{ grains}$. Therefore, the entire quotient is 6 pounds 1 ounce 8 pennyweights 6 grains. Ans.

EXAMPLE.—A silversmith melted up 2 lb. 8 oz. 10 pwt. of silver, which he made into 6 spoons; what was the weight of each spoon?

SOLUTION.—

	lb.	oz.	pwt.	
6) 2	8	10		
	5 oz. 8 pwt. 8 gr.			Ans.

EXPLANATION.—Since we can not divide 2 pounds by 6, we reduce it to ounces. 2 pounds = 24 ounces, and 24 ounces + 8 ounces = 32 ounces; 32 ounces $\div$ 6 = 5 ounces and 2 ounces over. 2 ounces = 40 pennyweights. 40 pennyweights + 10 pennyweights = 50 pennyweights, and 50 pennyweights $\div$ 6 = 8 pennyweights and 2 pennyweights over. 2 pennyweights = 48 grains, and 48 grains $\div$ 6 = 8 grains. Hence, each spoon contains 5 ounces 8 pennyweights 8 grains. Ans.

231. EXAMPLE.—Divide 820 rd. 4 yd. 2 ft. by 112.

	rd.	yd.	ft.	rd.	yd.	ft.	in.			
SOLUTION.—	112)	820	4	2	(	7	1	2	5.143	Ans.
		784								
			36	rd.	rem.					
			5.5							
			180							
			180							
			198.0	yd.						
			4							
	112)	202	yd.	(	1	yd.				
		112								
			90	yd.	rem.					
			3							
			270	ft.						
			2	ft.						
	112)	272	ft.	(	2	ft.				
		224								
			48	ft.	rem.					
			12							
			96							
			48							
	112)	576	in.	(	5.1428	+	in.,	or	5.143	in.
		560								
			160							
			112							
			480							
			448							
			320							
			224							
			960							
			896							
			64							

EXPLANATION.—The first quotient is 7 rods with 36 rods remaining. $5.5 \times 36 = 198$ yards; 198 yards + 4 yards = 202 yards; 202 yards $\div 112 = 1$ yard and 90 yards remaining. $90 \times 3 = 270$ feet; 270 feet + 2 feet = 272 feet; 272 feet $\div 112 = 2$ feet and 48 feet remaining; $48 \times 12 = 576$ inches; 576 inches $\div 112 = 5.143$ inches, nearly. Ans.

The preceding example is solved by long division, because the numbers are too large to deal with mentally. Instead of expressing the last result as a decimal, it might have been expressed as a common fraction. Thus, $576 \div 112 = 5\frac{18}{14} = 5\frac{9}{7}$ inches. The chief advantage of using a common fraction is that if the quotient be multiplied by the divisor, the result will always be the same as the original dividend.

232. Rule 34.—*Find how many times the divisor is contained in the first or highest denomination of the dividend. Reduce the remainder (if any) to the next lower denomination, and add to it the number in the given dividend expressing that denomination. Divide this new dividend by the divisor. The quotient will be the next denomination in the quotient required. Continue in this manner until the lowest denomination is reached. The successive quotients will constitute the entire quotient.*

EXAMPLES FOR PRACTICE.

233. Divide

(a) 376 mi. 276 rd. by 22; (b) 1,137 bu. 3 pk. 4 qt. 1 pt. by 10; (c) 84 cwt. 48 lb. 49 oz. by 16; (d) 78 sq. yd. 18 sq. ft. 41 sq. in. by 18; (e) 148 mi. 64 rd. 24 yd. by 12; (f) 100 tons 16 cwt. 18 lb. 11 oz. by 15; (g) 36 lb. 18 oz. 18 pwt. 14 gr. by 8; (h) 112 mi. 48 rd. by 100.

Ans. { (a) 17 mi. $41\frac{7}{11}$ rd.
 (b) 113 bu. 3 pk. 1 qt. $\frac{1}{2}$ pt.
 (c) 5 cwt. 28 lb. $3\frac{1}{8}$ oz.
 (d) 4 sq. yd. 4 sq. ft. $2\frac{5}{8}$ sq. in.
 (e) 12 mi. 112 rd. 2 yd.
 (f) 6 tons 14 cwt. 41 lb. $3\frac{1}{2}$ oz.
 (g) 4 lb. 8 oz. 7 pwt. $7\frac{1}{2}$ gr.
 (h) 1 mi. $38\frac{4}{5}$ rd.

INVOLUTION.

234. Involution is the process of multiplying a number by itself one or more times. The product obtained by multiplying a number by itself is called a **power** of that number.

Thus, the *second* power of 3 is 9, since 3×3 are 9.

The *third* power of 3 is 27, since $3 \times 3 \times 3$ are 27.

The *fifth* power of 2 is 32, since $2 \times 2 \times 2 \times 2 \times 2$ are 32.

235. An **exponent** is a *small figure* placed to the *right* and a little *above* a number to show to what *power* it is to be raised, or how many times the number is to be used as a factor, as the small figures ³, ⁵, and ² below:

$$3^3 = 3 \times 3 \times 3 = 27.$$

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243.$$

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

236. The **root** of a number is that number which, used the required number of times as a factor, produces the number. In the above cases, 3 is a root of 27, since $3 \times 3 \times 3$ are 27. It is also a root of 32, since $2 \times 2 \times 2 \times 2 \times 2$ are 32.

237. The *second* power of a number is called its **square**.

Thus, 5² is called the **square** of 5, or 5 *squared*, and its value is $5 \times 5 = 25$.

238. The *third* power of a number is called its **cube**.

Thus, 5³ is called the *cube* of 5, or 5 *cubed*, and its value is $5 \times 5 \times 5 = 125$.

To find any power of a number :

239. EXAMPLE.—What is the third power, or cube, of 35?

SOLUTION.—

$$\begin{array}{r}
 35 \times 35 \times 35, \\
 \text{or} \quad \begin{array}{r}
 35 \\
 35 \\
 \hline
 175 \\
 105 \\
 \hline
 1225 \\
 35 \\
 \hline
 6125 \\
 3675 \\
 \hline
 \text{cube} = 42875 \quad \text{Ans.}
 \end{array}
 \end{array}$$

240. EXAMPLE.—What is the fourth power of 15?

SOLUTION.— $15 \times 15 \times 15 \times 15$,

$$\begin{array}{r}
 \text{or} \quad 15 \\
 \quad 15 \\
 \quad \hline
 \quad 75 \\
 \quad 15 \\
 \quad \hline
 \quad 225 \\
 \quad 15 \\
 \quad \hline
 \quad 1125 \\
 \quad 225 \\
 \quad \hline
 \quad 3375 \\
 \quad 15 \\
 \quad \hline
 \quad 16875 \\
 \quad 3375 \\
 \quad \hline
 \quad 50625
 \end{array}$$

fourth power = 50625 Ans.

241. EXAMPLE.— $1.2^3 =$ what?

SOLUTION.—

$$1.2 \times 1.2 \times 1.2,$$

$$\begin{array}{r}
 \text{or} \quad 1.2 \\
 \quad 1.2 \\
 \quad \hline
 \quad 1.44 \\
 \quad 1.2 \\
 \quad \hline
 \quad 2.88 \\
 \quad 1.44 \\
 \quad \hline
 \quad 1.728
 \end{array}$$

1.728 Ans.

242. EXAMPLE.—What is the third power, or cube, of $\frac{3}{8}$?

$$\text{SOLUTION.—} \quad \left(\frac{3}{8}\right)^3 = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{3 \times 3 \times 3}{8 \times 8 \times 8} = \frac{27}{512} \text{ Ans.}$$

243. Rule 35.—(a) *To raise a whole number or a decimal to any power, use it as a factor as many times as there are units in the exponent.*

(b) *To raise a fraction to any power, raise both the numerator and denominator to the power indicated by the exponent.*

EXAMPLES FOR PRACTICE.

244. Raise the following to the powers indicated:

$$\begin{array}{ll}
 (a) \quad 85^3. & \\
 (b) \quad \left(\frac{11}{13}\right)^3. & \\
 (c) \quad 6.5^2. & \\
 (d) \quad 14^4. & \end{array}
 \quad \text{Ans.} \quad \left\{ \begin{array}{ll} (a) \quad 7,225. \\ (b) \quad \frac{1331}{2197}. \\ (c) \quad 42.25. \\ (d) \quad 38,416. \end{array} \right.$$

(e) $(\frac{1}{2})^2$.	Ans. {	(e) $\frac{1}{4}$.
(f) $(\frac{1}{3})^3$.		(f) $\frac{1}{27}$.
(g) $(\frac{1}{4})^3$.		(g) $\frac{1}{64}$.
(h) 1.4^5 .		(h) 5.37824.

EVOLUTION.

245. Evolution is the reverse of involution. It is the process of finding the root of a number which is considered as a power.

246. The **square root** of a number is that number which, when used twice as a factor, produces the number.

Thus, 2 is the square root of 4, since 2×2 , or $2^2 = 4$.

247. The **cube root** of a number is that number which, when used three times as a factor, produces the number.

Thus, 3 is the cube root of 27, since $3 \times 3 \times 3$, or $3^3 = 27$.

248. The **radical sign** $\sqrt{}$, when placed before a number, indicates that some root of that number is to be found.

249. The **index** of the root is a *small figure* placed *over* and to the *left* of the *radical sign*, to show what root is to be found.

Thus, $\sqrt[2]{100}$ denotes the *square root* of 100.

$\sqrt[3]{125}$ denotes the *cube root* of 125.

$\sqrt[4]{256}$ denotes the *fourth root* of 256, and so on.

250. When the square root is to be extracted, the index is generally omitted. Thus, $\sqrt{100}$ indicates the square root of 100. Also, $\sqrt{225}$ indicates the square root of 225.

SQUARE ROOT.

251. The *largest* number that can be written with *one* figure is 9, and $9^2 = 81$; the *largest* number that can be written with *two* figures is 99, and $99^2 = 9,801$; with *three* figures 999, and $999^2 = 998,001$; with *four* figures 9,999, and $9,999^2 = 99,980,001$, etc.

In *each* of the above it will be noticed that the square of the number contains just *twice* as many figures as the number.

In order to find the square root of a number, the first step is to find how many figures there will be in the root. This

is done by pointing off the number into *periods* of *two* figures each, *beginning at the right*. The number of periods will indicate the number of figures in the root.

Thus, the square root of 83,740,801 must contain 4 figures, since, pointing off the periods, we get 83'74'08'01, or 4 periods; consequently, there must be 4 figures in the root. In like manner, the square root of 50,625 must contain 3 figures, since there are (5'06'25) 3 periods.

252. EXAMPLE.—Find the square root of 31,505,769.

SOLUTION.— (a) $\begin{array}{r} 5 \\ \hline 5 \\ \hline 100 \\ \hline 6 \\ \hline 106 \\ \hline 6 \\ \hline 1120 \\ \hline 1 \\ \hline 1121 \\ \hline 1 \\ \hline 11220 \\ \hline 8 \\ \hline 11228 \end{array}$	$\begin{array}{r} 31'50'57'69 \text{ (} \overset{\text{root.}}{5613} \text{ Ans.} \\ \hline 25 \\ \hline 650 \\ \hline 636 \\ \hline 1457 \\ \hline 1121 \\ \hline 33669 \\ \hline 33669 \\ \hline 0 \end{array}$
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EXPLANATION.—Pointing off into periods of two figures each, it is seen that there are four figures in the root. Now, find the largest single number whose square is less than or equal to 31, the first period. This is evidently 5, since $6^2 = 36$, which is greater than 31. Write it to the right, as in long division, and also to the left, as shown at (a). This is the first figure of the root. Now, multiply the 5 at (a) by the 5 in the root, and write the result under the first period, as shown at (b). Subtract, and obtain 6 as a remainder.

Bring down the next period 50, and annex it to the remainder 6, as shown at (c), which we call the **dividend**. Add the root already found to the 5 at (a), getting 10, and annex a cipher to this 10, thus making it 100, which we call the **trial divisor**. Divide the dividend (c) by the trial divisor (d), and obtain 6, which is *probably* the next figure of the root. Write 6 in the root, as shown, and also add it

to 100, the trial divisor, making it 106. This is called the **complete divisor**.

Multiply this by 6, the second figure in the root, and subtract the result from the dividend (*c*). The remainder is 14, to which annex the next period, making it 1,457, as shown at (*e*), which we call the **new dividend**. Add the second figure of the root to the trial divisor 106, and annex a cipher, thus getting 1,120. Dividing 1,457 by 1,120, we get 1 as the next figure of the root. Adding this last figure of the root to 1,120, multiplying the result by it, and subtracting from 1,457, the remainder is 336.

Annexing the next and last period, 69, the result is 33,669. Now, adding the last figure of the root to 1,121, and annexing a cipher as before, the result is 11,220. Dividing 33,669 by 11,220, the result is 3, the fourth figure in the root. Adding it to 11,220, and multiplying the sum by it, the result is 33,669. Subtracting, there is no remainder; hence, $\sqrt[4]{31,505,769} = 5,613$. Ans.

253. The square of any number wholly decimal always contains twice as many figures as the number squared. For example, $.1^2 = .01$; $.13^2 = .0169$; $.751^2 = .564001$, etc.

254. It will also be noticed that the number squared is always less than the decimal. Hence, if it be required to find the square root of a decimal, and the decimal has not an even number of figures in it, annex a cipher. The best way to determine the number of figures in the root of a decimal is to begin at the decimal point, and, going towards the *right*, point off the decimal into periods of two figures each. Then, if the last period contains but one figure, annex a cipher.

255. EXAMPLE.—What is the square root of .000576?

SOLUTION.—	2	.00'05'76	Ans.
	2	4	
	40	176	
	4	176	
	44	0	

EXPLANATION.—Beginning at the decimal point, and pointing off the number into periods of two figures each, it is seen that the first period is composed of ciphers; hence, the first figure of the root must be a cipher. The remaining portion of the solution should be perfectly clear from what has preceded.

256. If the number is not a perfect power, the root will consist of an interminable number of decimal places. The result may be carried to any required number of decimal places by annexing periods of two ciphers each to the number.

257. EXAMPLE.—What is the square root of 3? Find the result to five decimal places.

SOLUTION. — 1 1 <u>20</u> 7 <u>27</u> 7 <u>340</u> 3 <u>343</u> 3 <u>3460</u> 2 <u>3462</u> 2 <u>346400</u> 5 <u>346405</u>	<i>root</i> 3.0000000000(173205 + Ans. 1 <u>200</u> 189 <u>1100</u> 1029 <u>7100</u> 6924 <u>1760000</u> 1732025 <u>27975</u>
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EXPLANATION.—Annexing five periods of two ciphers each to the right of the decimal point, the first figure of the root is 1. To get the second figure, we find that, in dividing 200 by 20, it is 10. This is evidently too large.

Trying 9, we add 9 to 20, and multiply 29 by 9, the result is 261, a result which is considerably larger than 200; hence,

9 is too large. In the same way it is found that 8 is also too large. Trying 7, 7 times 27 is 189, a result smaller than 200; therefore, 7 is the second figure of the root. The next two figures, 3 and 2, are easily found. The fifth figure in the root is a cipher, since the trial divisor 34,640 is greater than the new dividend 17,600. In a case of this kind, we annex another cipher to 34,640, thereby making it 346,400, and bring down the next period, making the 17,600, 1,760,000. The next figure of the root is 5, and as we now have five decimal places, we will stop.

The square root of 3 is, then, 1.73205 +. Ans.

258. EXAMPLE.—What is the square root of .3 to five decimal places?

SOLUTION.— $ \begin{array}{r} 5 \\ \hline 100 \\ 4 \\ \hline 104 \\ 4 \\ \hline 1080 \\ 7 \\ \hline 1087 \\ 7 \\ \hline 10940 \\ 7 \\ \hline 10947 \\ 7 \\ \hline 109540 \\ 2 \\ \hline 109542 \end{array} $	$ \begin{array}{r} .30'00'00'00'00'00' \overset{\text{root}}{.54772} + \text{Ans.} \\ \hline 25 \\ \hline 500 \\ 416 \\ \hline 8400 \\ 7609 \\ \hline 79100 \\ 76629 \\ \hline 247100 \\ 219084 \\ \hline 28016 \end{array} $
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EXPLANATION.—In the above example, we annex a cipher to .3, making the first period .30, since every period of a decimal, as was mentioned before, must have two figures in it. The remainder of the work should be perfectly clear.

259. If it is required to find the square root of a mixed number, begin at the decimal point, and point off the

periods both ways. The manner of finding the root will then be exactly the same as in the previous cases.

260. EXAMPLE.—What is the square root of 258.2449?

SOLUTION.—

1	258.2449 (16.07 Ans.
<u>1</u>	<u>1</u>
20	158
<u>6</u>	<u>156</u>
26	22449
<u>6</u>	<u>22449</u>
3200	0
<u>7</u>	
3207	

EXPLANATION.—In the above example, since 320 is greater than 224, we place a cipher for the third figure of the root, and annex a cipher to 320, making it 3,200. Then, bringing down the next period 49, 7 is found to be the fourth figure of the root. Since there is no remainder, the square root of 258.2449 is 16.07. Ans.

261. Proof.—*To prove square root, square the result obtained. If the number is an exact power, the square of the root will equal it; if it is not an exact power, the square of the root will very nearly equal it.*

262. Rule 36.—(a) *Begin at units place, and separate the number into periods of two figures each, proceeding from left to right with the decimal part, if there is any.*

(b) *Find the greatest number whose square is contained in the first or left-hand period. Write this number as the first figure in the root; also, write it at the left of the given number.*

Multiply this number at the left by the first figure of the root, and subtract the result from the first period; then annex the second period to the remainder.

(c) *Add the first figure of the root to the number in the first column on the left, and annex a cipher to the result; this is the trial divisor. Divide the dividend by the trial divisor*

for the second figure in the root, and add this figure to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure in the root, and subtract this result from the dividend. (If this result is larger than the dividend, a smaller number must be tried for the second figure of the root.) Now bring down the third period, and annex it to the last remainder for a new dividend. Add the second figure of the root to the complete divisor, and annex a cipher for a new trial divisor.

(d) Continue in this manner to the last period, after which, if any additional places in the root are required, bring down cipher periods, and continue the operation.

(e) If at any time the trial divisor is not contained in the dividend, place a cipher in the root, annex a cipher to the trial divisor, and bring down another period.

(f) If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.

263. Short Method.—If the number whose root is to be extracted is not an exact square, the root will be an interminable decimal. It is then usual to extract the root to a certain number of decimal places. In such cases, the work may be greatly shortened as follows: Determine to how many decimal places the work is to be carried, say 5, for example; add to this the number of places in the integral part of the root, say 2, for example, thus determining the number of figures in the root, in this case $5 + 2 = 7$. Divide this number by 2 and take the next higher number. In the above case, we have $7 \div 2 = 3\frac{1}{2}$; hence, we take 4, the next higher number. Now extract the root in the usual manner until the same number of figures have been obtained as was expressed by the number obtained above, in this case 4. Then form the trial divisor in the usual manner, but omitting to annex the cipher; divide the last remainder by the trial

divisor, as in long division, obtaining as many figures of the quotient as there are remaining figures of the root, in this case $7 - 4 = 3$. The remainder so obtained is the remaining figures of the root.

Consider the example in Art. 258. Here there are 5 figures in the root. We therefore extract the root to 3 places in the usual manner, obtaining .547 for the first three root figures. The next trial divisor is 1,094 (with the cipher omitted), and the last remainder is 791. Then, $791 \div 1,094 = .723$, and the next two figures of the root are 72, the whole root being .54772+. Always carry the division one place further than desired, and if the last figure is 5 or greater, increase the preceding figure by 1. This method should not be used unless the root contains five or more figures.

NOTE.—If the last figure of the root found in the regular manner is a cipher, carry the process one place further before dividing as described above.

EXAMPLES FOR PRACTICE.

264. Find the square root of

(a) 186,624.	Ans. {	(a) 432.
(b) 2,050,624.		(b) 1,432.
(c) 29,855,296.		(c) 5,464.
(d) .0116964.		(d) .1081+.
(e) 198.1369.		(e) 14.0761.
(f) 994,009.		(f) 997.
(g) 2.375 to four decimal places.		(g) 1.5411.
(h) 1.625 to three decimal places.		(h) 1.275.
(i) .3025.		(i) .55.
(j) .571428.		(j) .7559+.
(k) .78125.		(k) .8839.

CUBE ROOT.

265. In the same manner as in the case of square root, it can be shown that the periods into which a number is divided, whose cube root is to be extracted, must contain

three figures, except that the first or left-hand period of a whole or mixed number may contain one, two, or three figures.

266. EXAMPLE.—What is the cube root of 875,741,853,696?

SOLUTION.—

(1)	(2)	(3)	<i>root</i>
7	49	375'741'853'696	(7216 Ans
7	98	343	
14	14700	32741	
7	424	30248	
210	15124	2493853	
2	428	1557361	
212	1555200	936492696	
2	2161	936492696	
214	1557361	0	
2	2162		
2160	155952300		
1	129816		
2161	156082116		
1			
2162			
1			
21630			
6			
21636			

EXPLANATION.—Write the work in three columns as follows: On the right place the number whose cube root is to be extracted, and point it off into periods of three figures each. Call this column (3). Find the largest number whose cube is less than or equal to the first period, in this case 7. Write the 7 on the right, as shown, for the first figure of the root, and also on the extreme left at the head of column (1). Multiply the 7 in column (1) by the first figure of the root 7, and write the product 49 at the head of column (2). Multiply the number in column (2) by the first figure of the root 7, and write the product 343 under the figures in the

first period. Subtract and bring down the next period, obtaining 32,741 for the dividend. Add the first figure of the root to the number in column (1), obtaining 14, which call the *first correction*. Multiply the first correction by the first figure of the root, add the product to the number in column (2), and obtain 147. Add the first figure of the root to the first correction, and obtain 21, which call the *second correction*. Annex *two* ciphers to the number in column (2), and obtain 14,700 for the trial divisor; also, annex *one* cipher to the second correction, and obtain 210. Dividing the dividend by the trial divisor, we obtain $\frac{32,741}{14,700} = 2 +$, and write the 2 as the second figure of the root. Add the 2 to the second correction, and obtain 212, which, multiplied by the second figure of the root, and added to the trial divisor, gives 15,124, the complete divisor. This last result, multiplied by the second figure of the root and subtracted from the dividend, gives a remainder of 2,493. Annexing the third period, we obtain 2,493,853 for the new dividend. Adding the second figure of the root to the number in column (1), we get 214 as the new first correction; this, multiplied by the second figure of the root and added to the trial divisor, gives 15,552. Adding the second figure of the root to the first new correction gives 216 as the second new correction. Annexing two ciphers to the number in column (2) gives 1,555,200, the new trial divisor. Annexing one cipher to the second new correction gives 2,160. Dividing the new dividend by the new trial divisor, we obtain $\frac{2,493,853}{1,555,200} = 1 +$, and write 1 as the third figure of the root. The remainder of the work should be perfectly clear from what has preceded.

267. In extracting the cube root of a decimal, proceed as above, taking care that each period contains *three* figures. Begin the pointing off at the decimal point, going towards the right. If the last period does not contain three figures, annex ciphers until it does.

268. EXAMPLE.—What is the cube root of .009129329?

SOLUTION.—	2	4	.009'129'329 (^{root} .209
	<u>2</u>	<u>8</u>	<u>8</u>
	4	120000	1129329
	<u>2</u>	<u>5481</u>	<u>1129329</u>
	600	125481	0
	<u>9</u>		
	609		

EXPLANATION.—Beginning at the decimal point, and pointing off as shown, the largest number whose cube is less than 9 is seen to be 2; hence, 2 is the first figure of the root. When finding the second figure, it is seen that the trial divisor 1,200 is greater than the dividend; hence, write a cipher for the second figure of the root; bring down the next period to form the new dividend; annex two ciphers to the trial divisor to form a new trial divisor; also, annex one cipher to the 60 in column (1). Dividing the new dividend by the new trial divisor, we get $\frac{1,129,329}{120,000} = 9 +$, and write 9 as the third figure of the root. Complete the work as before.

269. EXAMPLE.—What is the cube root of 78,292.892952?

SOLUTION.—			^{root}
	4	16	78'292.892'952 (42.78
	<u>4</u>	<u>82</u>	<u>64</u>
	8	4800	14292
	<u>4</u>	<u>244</u>	<u>10088</u>
	120	5044	4204892
	<u>2</u>	<u>248</u>	<u>3766483</u>
	122	529200	438409952
	<u>2</u>	<u>8869</u>	<u>438409952</u>
	124	538069	0
	<u>2</u>	<u>8918</u>	
	1260	54698700	
	<u>7</u>	<u>102544</u>	
	1267	54801244	
	<u>7</u>		
	1274		
	<u>7</u>		
	12810		
	<u>8</u>		
	12818		

EXPLANATION.—Since we have a mixed number, begin at the decimal point and point off periods of three figures each, in both directions. The first period contains but two figures, and the largest number whose cube is less than 78 is 4; consequently, 4 is the first figure of the root. The remainder of the work should be perfectly clear. When dividing the dividend by the trial divisor for the third figure of the root, the quotient was 8 +; but, on trying it, it was found that 8 was too large, the complete divisor being considerably larger than the trial divisor. Therefore, 7 was used instead of 8.

270. EXAMPLE.—What is the cube root of 5 to five decimal places?

SOLUTION.—

			<i>root</i>
1	1	5.000'000'000'000'000	(1.70997 +
1	2	1	
2	300	4000	
1	259	3918	
30	559	87000000	
7	308	78443829	
37	8670000	8556171000	
7	45981	7889992299	
44	8715981	666178701000	
7	46062	614014317973	
5100	876204300	52164383027	
9	461511		
5109	876665811		
9	461592		
5118	87712740300		
9	3590839		
51270	87716331139		
9			
51279			
9			
51288			
9			
512970			
7			
512977			

EXPLANATION.—In the preceding example we annex five periods of ciphers, of three ciphers each, to the 5 for the decimal part of the root, placing the decimal point between the 5 and the first cipher. Since it is easy to see that the next figure of the root will be 5, we increase the last figure by 1, obtaining 1.70998 for the correct root to 5 decimal places. **Ans.**

271. EXAMPLE.—What is the cube root of .5 to four decimal places?

SOLUTION.—

7	49	.500'000'000'000'000 (^{root} .7937 +
7	98	343
14	14700	157000
7	1971	150039
210	16671	6961000
9	2052	5638257
219	1872300	1322743000
9	7119	1321748953
228	1879419	994047
9	7128	
2370	188654700	
3	166579	
2373	188821279	
8		
2376		
8		
23790		
7		
23797		

EXPLANATION.—In the above example we annex two ciphers to the .5 to complete the first period, and three periods of three ciphers each. The cube root of 500 is 7; this we write as the first figure of the root. The remainder of the work should be perfectly plain from the explanations of the preceding examples.

272. EXAMPLE.—What is the cube root of .05 to four decimal places?

SOLUTION.—

		<i>root</i>
3	9	.050'000'000'000 (.3684 +
3	18	27
6	2700	23000
3	576	19656
90	3276	3344000
6	612	3180032
96	388800	163968000
6	8704	162685504
102	697504	1282496
6	8768	
1080	40627200	
8	44176	
1088	40671376	
8		
1096		
8		
11040		
4		
11044		

273. Proof.—To prove cube root, cube the result obtained. If the given number is an exact power, the cube of the root will equal it; if not an exact power, the cube of the root will very nearly equal it.

274. Rule 37.—(a) Arrange the work in three columns, placing the number whose cube root is to be extracted in the third or right-hand column. Begin at units place, and separate the number into periods of three figures each, proceeding from the decimal point towards the right with the decimal part, if there is any.

(b) Find the greatest number whose cube is not greater than the number in the first period. Write this number as the first figure of the root; also, write it at the head of the first column. Multiply the number in the first column by the first figure in the root, and write the result in the second

column. Multiply the number in the second column by the first figure of the root; subtract the product from the first period, and annex the second period to the remainder for a new dividend; add the first figure of the root to the number in the first column for the first correction. Multiply the first correction by the first figure of the root, and add the product to the number in the second column. Add the first figure of the root to the first correction to form the second correction. Annex one cipher to the second correction, and two ciphers to the last number in the second column; the last number in the second column is the trial divisor.

(c) Divide the dividend by the trial divisor to find the second figure of the root. Add the second figure of the root to the number in the first column, multiply the sum by the second figure of the root, and add the result to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure of the root, subtract the result from the dividend in the third column, and annex the third period to the remainder for a new dividend. Add the second figure of the root to the number in the first column to form the first correction; multiply the first correction by the second figure of the root, and add the product to the complete divisor. Add the second figure of the root to the first correction to form the second correction. Annex one cipher to the second correction, and two ciphers to the last number in the second column to form the new trial divisor.

(d) If there are more periods to be brought down, proceed as before. If there is a remainder after the root of the last period has been found, annex cipher periods, and proceed as before. The figures of the root thus obtained will be decimals.

(e) If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.

Art. 263 can be applied to cube root (or any other root) as well as to square root. Thus, in the example,

Art. 270, there are to be $5 + 1 = 6$ figures in the root. Extracting the root in the usual manner to $6 \div 2 = 3$, say 4 figures, we get for the first four figures 1,709. The last remainder is 8,556,171, and the next trial divisor with the ciphers omitted is 8,762,043. Hence, the next two figures of the root are $8,556,171 \div 8,762,043 = .976$, say .98. Therefore, the root is 1.70998.

ROOTS OF FRACTIONS.

275. If a given number is in the form of a fraction and it is required to find some root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the required root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the required root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

276. EXAMPLE.—What is the square root of $\frac{9}{64}$?

$$\text{SOLUTION.}—\sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}.$$

277. EXAMPLE.—What is the square root of $\frac{625}{64}$?

$$\text{SOLUTION.}—\sqrt{\frac{625}{64}} = \frac{\sqrt{625}}{\sqrt{64}} = \frac{25}{8} = 3.125.$$

278. EXAMPLE.—What is the cube root of $\frac{27}{64}$?

$$\text{SOLUTION.}—\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}.$$

279. EXAMPLE.—What is the cube root of $\frac{1}{8}$?

$$\text{SOLUTION.}—\text{Since } \frac{1}{8} = .125, \sqrt[3]{\frac{1}{8}} = \sqrt[3]{.125} = .5.$$

280. Rule 38.—*Extract the required root of the numerator and denominator separately; or, reduce the fraction to a decimal and extract the root of the decimal.*

EXAMPLES FOR PRACTICE.

281. Find the cube root of

- | | |
|--|---|
| (a) $\frac{27}{125}$. | Ans. $\left\{ \begin{array}{l} (a) \frac{3}{5}. \\ (b) 1.2599+. \\ (c) 1,610.962. \\ (d) .8862+. \\ (e) .7211+. \\ (f) 80.064. \end{array} \right.$ |
| (b) 2 to four decimal places. | |
| (c) 4,180,769,192.462 to three decimal places. | |
| (d) $\frac{87}{125}$. | |
| (e) $\frac{8}{8}$. | |
| (f) 513,229.783302144 to three decimal places. | |

TO EXTRACT OTHER ROOTS THAN THE SQUARE AND CUBE ROOTS.

282. EXAMPLE.—What is the fourth root of 256?

SOLUTION.—

$$\sqrt[4]{256} = 16.$$

$$\sqrt{16} = 4.$$

Therefore,

$$\sqrt[4]{256} = 4. \text{ Ans.}$$

In this example, $\sqrt[4]{256}$, the index is 4, which equals 2×2 . The root indicated by 2 is the square root; therefore, the square root is extracted twice.

283. EXAMPLE.—What is the sixth root of 64?

SOLUTION.—

$$\sqrt{64} = 8.$$

$$\sqrt[3]{8} = 2.$$

Therefore,

$$\sqrt[6]{64} = 2. \text{ Ans.}$$

In this example, $\sqrt[6]{64}$, the index is 6, which equals 2×3 . The root indicated by 3 is the cube root; therefore, the square and cube roots are extracted in succession.

284. Rule 39.—*Separate the index of the required root into its factors (2's and 3's), and extract successively the roots indicated by the several factors obtained. The final result will be the required root.*

285. EXAMPLE.—What is the sixth root of 92,873,580 to two decimal places?

SOLUTION.— $6 = 3 \times 2$. Hence, extract the cube root, and then extract the square root of the result. $\sqrt[3]{92,873,580} = 452.8601$, and $\sqrt{452.8601} = 21.28 +$. Ans.

286. It matters not which root is extracted first, but it is probably easier and more exact to extract the cube root first.

EXAMPLES FOR PRACTICE.

287. Extract the

(a) Fourth root of 100.

(b) Fourth root of 3,049,800,625.

(c) Sixth root of 9,474,296,896.

Ans. $\left\{ \begin{array}{l} (a) \ 3.16227+. \\ (b) \ 235. \\ (c) \ 46. \end{array} \right.$

RATIO.

288. Suppose that it is desired to compare two numbers, say 20 and 4. If we wish to know how many times larger 20 is than 4, we divide 20 by 4 and obtain 5 for the quotient; thus, $20 \div 4 = 5$. Hence, we say that 20 is 5 times as large as 4, i. e., 20 contains 5 times as many units as 4. Again, suppose we desire to know what part of 20 is 4. We then divide 4 by 20 and obtain $\frac{1}{5}$; thus, $4 \div 20 = \frac{1}{5}$, or .2. Hence, 4 is $\frac{1}{5}$ or .2 of 20. This operation of comparing two numbers is termed *finding the ratio* of the two numbers. Ratio, then, is a comparison. It is evident that the two numbers to be compared must be expressed in the same unit; in other words, the two numbers must both be abstract numbers or concrete numbers of the same kind. For example, it would be absurd to compare 20 horses with 4 birds, or 20 horses with 4. Hence, **ratio** may be defined as a comparison between two numbers of the same kind.

289. A ratio may be *expressed* in three ways; thus, if it is desired to compare 20 and 4, and express this comparison as a ratio, it may be done as follows: $20 \div 4$; $20 : 4$, or $\frac{20}{4}$. All three are read *the ratio of 20 to 4*. The ratio of

4 to 20 would be expressed thus: $4 \div 20$; $4 : 20$, or $\frac{4}{20}$.

The first method of expressing a ratio, although correct, is seldom or never used; the second form is the one oftenest met with, while the third is rapidly growing in favor, and is likely to supersede the second. The third form, called the fractional form, is preferred by modern mathematicians, and possesses great advantages to students of algebra and of higher mathematical subjects. The second form seems to be better adapted to arithmetical subjects, and is one we shall ordinarily adopt. There is still another way of expressing a ratio, though seldom or never used in the case of a simple ratio like that given above. Instead of the colon, a straight vertical line is used; thus, $20 \mid 4$.

290. The **terms** of a ratio are the two numbers to be compared; thus, in the above ratio, 20 and 4 are the terms. When both terms are considered together, they are called a **couplet**; when considered separately, the first term is called the **antecedent**, and the second term the **consequent**. Thus, in the ratio 20 : 4, 20 and 4 form a couplet, and 20 is the antecedent, and 4 the consequent.

291. A ratio may be **direct** or **inverse**. The *direct ratio* of 20 to 4 is 20 : 4, while the *inverse ratio* of 20 to 4 is 4 : 20. The direct ratio of 4 to 20 is 4 : 20, and the inverse ratio is 20 : 4. An inverse ratio is sometimes called a **reciprocal** ratio. The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 17 is $\frac{1}{17}$; of $\frac{8}{9}$ is $1 \div \frac{8}{9} = \frac{9}{8}$; i.e., the reciprocal of a fraction is the fraction inverted. Hence, the inverse ratio of 20 to 4 may be expressed as 4 : 20, or as $\frac{1}{20} : \frac{1}{4}$. Both have equal values; for, $4 \div 20 = \frac{1}{5}$, and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{20} \times \frac{4}{1} = \frac{1}{5}$.

292. The term **vary** implies a ratio. When we say that two numbers vary as some other two numbers, we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

293. The **value** of a ratio is the result obtained by performing the division indicated. Thus, the value of the ratio 20 : 4 is 5, it is the quotient obtained by dividing the antecedent by the consequent.

294. By expressing the ratio in the fractional form, for example, the ratio of 20 to 4 as $\frac{20}{4}$, it is easy to see, from the laws of fractions, that if both terms be multiplied, or both divided by the same number, it will not alter the value of the ratio. Thus,

$$\frac{20}{4} = \frac{20 \times 5}{4 \times 5} = \frac{100}{20}; \text{ and } \frac{20}{4} = \frac{20 \div 4}{4 \div 4} = \frac{5}{1}.$$

295. It is also evident, from the laws of fractions, that multiplying the antecedent or dividing the consequent multiplies the ratio; and dividing the antecedent or multiplying the consequent divides the ratio.

296. When a ratio is expressed in words, as the ratio of 20 to 4, the first number named is always regarded as the antecedent and the second as the consequent, without regard to whether the ratio itself is direct or inverse. *When not otherwise specified, all ratios are understood to be direct.* To express an inverse ratio, the simplest way of doing it is to express it as if it were a direct ratio, with the first number named as the antecedent, and then transpose the antecedent to the place occupied by the consequent and the consequent to the place occupied by the antecedent; or if expressed in the fractional form, invert the fraction. Thus, to express the inverse ratio of 20 to 4, first write it 20 : 4, and then, transposing the terms, as 4 : 20; or as $\frac{20}{4}$, and then inverting as $\frac{4}{20}$. Or, the reciprocals of the numbers may be taken, as explained above. To **invert** a ratio is to transpose its terms.

EXAMPLES FOR PRACTICE.

297. What is the value of the ratio of

(a) 98 to 49 ?	Ans. {	(a) 2.
(b) \$45 to \$9 ?		(b) 5.
(c) $6\frac{1}{2}$ to $\frac{2}{3}$?		(c) $12\frac{1}{2}$.
(d) 3.5 to 4.5 ?		(d) $.77\frac{1}{2}$.
(e) The inverse ratio of 76 to 19 ?		(e) $\frac{1}{4}$.
(f) The inverse ratio of 49 to 98 ?		(f) 2.
(g) The inverse ratio of 18 to 24 ?		(g) $1\frac{1}{2}$.
(h) The inverse ratio of 9 to 15 ?		(h) $1\frac{1}{3}$.
(i) The ratio of 10 to 3, multiplied by 3 ?		(i) 10.
(j) The ratio of 35 to 49, multiplied by 7 ?		(j) 5.
(k) The ratio of 18 to 64, divided by 9 ?		(k) $\frac{1}{8}$.
(l) The ratio of 14 to 28, divided by 5 ?		(l) $\frac{1}{10}$.

298. Instead of expressing the value of a ratio by a single number, as above, it is customary to express it by

means of another ratio in which the consequent is 1. Thus, suppose that it is desired to find the ratio of the weights of two pieces of iron, one weighing 45 pounds and the other weighing 30 pounds. The ratio of the heavier to the lighter is then $45 : 30$, an inconvenient expression. Using the fractional form, we have $\frac{45}{30}$. Dividing both terms by 30, the consequent, we obtain $\frac{1\frac{1}{2}}{1}$ or $1\frac{1}{2} : 1$. This is the same result as obtained above, for $1\frac{1}{2} \div 1 = 1\frac{1}{2}$, and $45 \div 30 = 1\frac{1}{2}$.

299. A ratio may be squared, cubed, or raised to any power, or any root of it may be taken. Thus, if the ratio of two numbers is $105 : 63$, and it is desired to cube this ratio, the cube may be expressed as $105^3 : 63^3$. That this is correct is readily seen; for, expressing the ratio in the fractional form, it becomes $\frac{105}{63}$, and the cube is $\left(\frac{105}{63}\right)^3 = \frac{105^3}{63^3} = 105^3 : 63^3$. Also, if it is desired to extract the cube root of the ratio $105^3 : 63^3$, it may be done by simply dividing the exponents by 3, obtaining $105 : 63$. This may be proved in the same way as in the case of cubing the ratio. Thus, $105^3 : 63^3 = \left(\frac{105}{63}\right)^3$, and $\sqrt[3]{\left(\frac{105}{63}\right)^3} = \frac{105}{63} = 105 : 63$.

300. Since $\left(\frac{105}{63}\right)^3 = \left(\frac{5}{3}\right)^3$, it follows that $105^3 : 63^3 = 5^3 : 3^3$ (this expression is read: the ratio of 105 cubed to 63 cubed equals the ratio of 5 cubed to 3 cubed), it follows that the antecedent and consequent may always be multiplied or divided by the same number, irrespective of any indicated powers or roots, without altering the value of the ratio. Thus, $24^3 : 18^3 = 4^3 : 3^3$. For, performing the operations indicated by the exponents, $24^3 = 576$ and $18^3 = 324$. Hence, $576 : 324 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $4^3 = 16$ and $3^3 = 9$; hence, $16 : 9 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$, the same result as before. Also, $24^3 : 18^3 = \frac{24^3}{18^3} = \left(\frac{24}{18}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3} = 4^3 : 3^3$.

The statement may be proved for roots in the same manner. Thus, $\sqrt[3]{24^3} : \sqrt[3]{18^3} = \sqrt[3]{4^3} : \sqrt[3]{3^3}$. For the $\sqrt[3]{24^3} = 24$ and $\sqrt[3]{18^3} = 18$; and, $24 : 18 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $\sqrt[3]{4^3} = 4$ and $\sqrt[3]{3^3} = 3$; $4 : 3 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$.

NOTE.—If the numbers composing the antecedent and consequent have different exponents, or if different roots of those numbers are indicated, the operations described in Art. 300 **can not** be performed. This is evident; for, consider the ratio $4^3 : 8^3$. When expressed in the fractional form, it becomes $\frac{4^3}{8^3}$, which **can not** be expressed either as $\left(\frac{4}{8}\right)^3$ or as $\left(\frac{4}{8}\right)^3$, and, hence, **can not** be reduced as described above.

PROPORTION.

301. Proportion is an equality of ratios, the equality being indicated by the double colon (::) or by the sign of equality (=). Thus, to write in the form of a proportion the two equal ratios, 8:4 and 6:3, which both have the same value 2, we may employ one of the three following forms:

$$8 : 4 :: 6 : 3 \quad (1)$$

$$8 : 4 = 6 : 3 \quad (2)$$

$$\frac{8}{4} = \frac{6}{3} \quad (3)$$

302. The first form is the one most extensively used, by reason of its having been exclusively employed in all the older works on mathematics. The second and third forms are being adopted by all modern writers on mathematical subjects, and, in time, will probably entirely supersede the first form. In this paper we shall adopt the second form, unless some statement can be made clearer by using the third form.

303. A proportion may be *read* in two ways. The old way to read the above proportion was: *8 is to 4 as 6 is to 3*; the new way is: *the ratio of 8 to 4 equals the ratio of 6 to 3*. The student may read it either way, but we recommend the latter.

304. Each ratio of a proportion is termed a **couplet**. In the above proportion, 8:4 is a couplet, and so is 6:3.

305. The numbers forming the proportion are called **terms**; and they are numbered consecutively from left to right, thus :

$$\begin{array}{cccc} \textit{first} & \textit{second} & \textit{third} & \textit{fourth} \\ 8 & : 4 = & 6 & : 3 \end{array}$$

Hence, in any proportion, the ratio of the first term to the second term equals the ratio of the third term to the fourth term.

306. The first and fourth terms of a proportion are called the **extremes**, and the second and third terms the **means**. Thus, in the foregoing proportion, 8 and 3 are the extremes and 4 and 6 are the means.

307. A **direct proportion** is one in which both couplets are direct ratios.

308. An **inverse proportion** is one which requires one of the couplets to be expressed as an inverse ratio. Thus, 8 is to 4 inversely as 3 is to 6 must be written $8 : 4 = 6 : 3$; i. e., the second ratio (couplet) must be inverted.

309. Proportion forms one of the most useful sections of arithmetic. In our grandfathers' arithmetics, it was called "The rule of three."

310. Rule 40.—*In any proportion, the product of the extremes equals the product of the means.*

Thus, in the proportion,

$$17 : 51 = 14 : 42.$$

$$17 \times 42 = 51 \times 14, \text{ since both products equal } 714.$$

311. Rule 41.—*The product of the extremes divided by either mean gives the other mean.*

EXAMPLE.—What is the third term of the proportion $17 : 51 = : 42$?

SOLUTION.—Applying the rule, $17 \times 42 = 714$, and $714 \div 51 = 14$. Ans.

312. Rule 42.—*The product of the means divided by either extreme gives the other extreme.*

EXAMPLE.—What is the first term of the proportion $: 51 = 14 : 42$?

SOLUTION.—Applying the rule, $51 \times 14 = 714$, and $714 \div 42 = 17$. Ans.

313. When stating a proportion in which one of the terms is unknown, represent the missing term by a letter, as x . Thus, the last example would be written,

$$x : 51 = 14 : 42$$

and for the value of x we have $x = \frac{51 \times 14}{42} = 17$.

314. If the same (addition and subtraction excepted) operations be performed upon *all* of the terms of a proportion, the proportion is not thereby destroyed. In other words, if all of the terms of a proportion be (1) multiplied or (2) divided by the same number; (3) if all the terms be raised to the same power; if (4) the same root of all the terms be taken, or (5) if both couplets be inverted, the proportion still holds. We will prove these statements by a numerical example, and the student can satisfy himself by other similar ones. The fractional form will be used, as it is better suited to the purpose. Consider the proportion $8 : 4 = 6 : 3$. Expressing it in the third form, it becomes $\frac{8}{4} = \frac{6}{3}$. What we are to prove is that, if any of the five operations enumerated above be performed upon all of the terms of this proportion, the first fraction will still equal the second fraction.

1. Multiplying all the terms by any number, say 7, $\frac{8 \times 7}{4 \times 7} = \frac{6 \times 7}{3 \times 7}$; or $\frac{56}{28} = \frac{42}{21}$. Now $\frac{56}{28}$ evidently equals $\frac{42}{21}$, since the value of either ratio is 2, and the same is true of the original proportion.

2. Dividing all the terms by any number, say 7, $\frac{8 \div 7}{4 \div 7} = \frac{6 \div 7}{3 \div 7}$; or $\frac{\frac{8}{7}}{\frac{4}{7}} = \frac{\frac{6}{7}}{\frac{3}{7}}$. But $\frac{8}{7} \div \frac{4}{7} = 2$, and $\frac{6}{7} \div \frac{3}{7} = 2$ also, the same as in the original proportion.

3. Raising all the terms to the same power, say the cube, $\frac{8^3}{4^3} = \frac{6^3}{3^3}$. This is evidently true, since $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8$, and $\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8$ also.

4. Extracting the same root of all the terms, say the cube root, $\sqrt[3]{\frac{8}{4}} = \frac{\sqrt[3]{8}}{\sqrt[3]{4}}$. It is evident that this is likewise true, since $\sqrt[3]{\frac{8}{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$, and $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$ also.

5. Inverting both couplets, $\frac{4}{8} = \frac{3}{6}$, which is true, since both equal $\frac{1}{2}$.

315. If both terms of either couplet be multiplied or both divided by the same number, the proportion is not destroyed. This should be evident from the preceding article, and also from Art. **294**. Hence, in any proportion, equal factors may be canceled from the terms of a couplet, before applying rule **41** or **42**. Thus, the proportion $45:9 = x:7.1$, we may divide both terms of the first couplet by 9 (that is, cancel 9 from both terms), obtaining $5:1 = x:7.1$, whence $x = 7.1 \times 5 \div 1 = 35.5$. (See note in Art. **300**.)

316. The principle of all calculations in proportion is this: *Three of the terms are always given, and the remaining one is to be found.*

317. EXAMPLE.—If 4 men can earn \$25 in one week, how much can 12 men earn in the same time?

SOLUTION.—The required term must bear the same relation to the given term of the same kind as one of the remaining terms bears to the other remaining term. We can then form a proportion by which the required term may be found.

The first question the student must ask himself in every calculation by proportion is:

“What is it I want to find?”

In this case it is dollars. We have two sets of men, one set earning \$25, and we want to know how many dollars the other set earns. It is evident that the *amount* 12 men earn bears the same relation to the *amount* that 4 men earn as 12 men bears to 4 men. Hence, we have the proportion, the amount 12 men earn is to \$25 as 12 men is to 4 men; or, since either extreme equals the product of the means divided by the other extreme, we have

The amount 12 men earn : \$25 = 12 men : 4 men,

or the amount 12 men earn = $\frac{\$25 \times 12}{4} = \$75$. Ans.

Since it matters not which place x or the required term occupies, the problem could be stated as any of the following forms, the value of x being the same in each :

(a) \$25 : the amount 12 men earn = 4 men : 12 men ; or the amount 12 men earn = $\frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

(b) 4 men : 12 men = \$25 : the amount 12 men earn ; or the amount that 12 men earn = $\frac{\$25 \times 12}{4}$, or \$75, since either extreme equals the product of the means divided by the other extreme.

(c) 12 men : 4 men = the amount 12 men earn : \$25 ; or the amount that 12 men earn = $\frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

318. If the proportion is an inverse one, first form it as though it were a direct proportion, and then invert one of the couplets.

EXAMPLES FOR PRACTICE.

319. Find the value of x in each of the following:

(a) \$16 : \$64 :: x : \$4.	Ans. {	(a) x = \$1.
(b) x : 85 :: 10 : 17.		(b) x = 50.
(c) 24 : x :: 15 : 40.		(c) x = 64.
(d) 18 : 94 :: 2 : x .		(d) x = 10 $\frac{1}{2}$.
(e) \$75 : \$100 = x : 100.		(e) x = 75.
(f) 15 pwt. : x = 21 : 10.		(f) x = 7 $\frac{1}{3}$ pwt.
(g) x : 75 yd. = \$15 : \$5.		(g) x = 225 yd.

1. If 75 pounds of lead cost \$2.10, what would 125 pounds cost at the same rate ? Ans. \$3.50.

2. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days ? Ans. 36 days.

3. The circumferences of any two circles are to each other as their diameters. If the circumference of a circle 7 inches in diameter is 22 inches, what will be the circumference of a circle 31 inches in diameter ? Ans. 97 $\frac{1}{3}$ inches.

INVERSE PROPORTION.

320. In Art. 308, an inverse proportion was defined as one which required one of the couplets to be expressed as an inverse ratio. Sometimes the word *inverse* occurs in the

statement of the example ; in such cases the proportion can be written directly, merely inverting one of the couplets. But it frequently happens that only by carefully studying the conditions of the example can it be ascertained whether the proportion is direct or inverse. When in doubt, the student can always satisfy himself as to whether the proportion is direct or inverse by first ascertaining what is required, and stating the proportion as a direct proportion. Then, in order that the proportion may be true, if the first term is smaller than the second term, the third term must be smaller than the fourth ; or if the first term is larger than the second term, the third term must be larger than the fourth term. Keeping this in mind, the student can always tell whether the required term will be larger or smaller than the other term of the couplet to which the required term belongs. Having determined this, the student then refers to the example, and ascertains from its conditions whether the required term is to be larger or smaller than the other term of the same kind. If the two determinations agree, the proportion is direct ; otherwise, it is inverse, and one of the couplets must be inverted.

321. EXAMPLE.—If A's *rate* of doing work is to B's as 5 : 7, and A does a piece of work in 42 days, in what time will B do it ?

SOLUTION.—The required term is the number of days it will take B to do the work. Hence, stating as a direct proportion,

$$5 : 7 = 42 : x.$$

Now, since 7 is greater than 5, x will be greater than 42. But, referring to the statement of the example, it is easy to see that B works faster than A ; hence it will take B a less number of days to do the work than A. Therefore, the proportion is an inverse one, and should be stated

$$5 : 7 = x : 42,$$

from which $x = \frac{5 \times 42}{7} = 30$ days. Ans.

Had the example been stated thus: The time that A requires to do a piece of work is to the time that B requires, as 5 : 7 ; A can do it in 42 days, in what time can B do it ? it is evident that it would take B a longer time to do the work than it would A ; hence, x would be greater than 42, and the proportion would be direct, the value of x being $\frac{7 \times 42}{5} = 58.8$ days.

EXAMPLES FOR PRACTICE.

322. Solve the following:

1. If a pump which discharges 4 gal. of water per min. can fill a tank in 20 hr., how long will it take a pump discharging 12 gal. per min. to fill it? Ans. $6\frac{2}{3}$ hr.

2. The circular seam of a boiler requires 50 rivets when the pitch is $2\frac{1}{4}$ in.; how many would be required if the pitch were $3\frac{1}{8}$ in.?

Ans. 40.

3. The spring hangers on a certain locomotive are $2\frac{1}{2}$ in. wide and $\frac{3}{4}$ in. thick; those on another engine are of same sectional area, but are 3 in. wide; how thick are they? Ans. $\frac{5}{8}$ in.

4. A locomotive with driving wheels 16 ft. in circumference runs a certain distance in 5,000 revolutions; how many revolutions would it make in going the same distance, if the wheels were 22 ft. in circumference (no allowance for slip being made in either case)?

Ans. $3,636\frac{4}{11}$ rev.

POWERS AND ROOTS IN PROPORTION.

323. It was stated in Art. 299 that a ratio could be raised to any power or any root of it might be taken. A proportion is frequently stated in such a manner that one of the couplets must be raised to some power or some root of it must be taken. In all such cases, both terms of the couplet so affected *must be raised to the same power or the same root of both terms must be taken.*

324. EXAMPLE.—Knowing that the weight of a sphere varies as the cube of its diameter, what is the weight of a sphere 6 inches in diameter if a sphere 8 inches in diameter of the same material weighs 180 pounds?

SOLUTION.—This is evidently a direct proportion. Hence, we write

$$6^3 : 8^3 = x : 180.$$

Dividing both terms of the first couplet by 2^3 (see Art. 300),

$$3^3 : 4^3 = x : 180, \text{ or } 27 : 64 = x : 180;$$

$$\text{whence, } x = \frac{27 \times 180}{64} = 75\frac{1}{8} \text{ pounds. Ans.}$$

EXAMPLE.—A sphere 8 inches in diameter weighs 180 pounds; what is the diameter of another sphere of the same material which weighs $75\frac{1}{8}$ pounds?

SOLUTION.—Since the weights of any two spheres are to each other as the cubes of their diameters, we have the proportion

$$180 : 75\frac{1}{8} = 8^3 : x^3.$$

x , the required term, must be cubed, because the other term of the couplet is cubed (see Art. 323). But, $8^3 = 512$; hence,

$$180 : 75\frac{1}{8} = 512 : x^3, \text{ or } x^3 = \frac{75\frac{1}{8} \times 512}{180} = 216;$$

whence, $x = \sqrt[3]{216} = 6$ inches. Ans.

325. Since taking the same root of all of the terms of a proportion does not change its value (Art. 314), the above example might have been solved by extracting the cube root of all of the numbers, thus obtaining $\sqrt[3]{180} : \sqrt[3]{75\frac{1}{8}} = 8 : x$; whence,

$$x = \frac{8 \times \sqrt[3]{75\frac{1}{8}}}{\sqrt[3]{180}} = 8 \times \frac{\sqrt[3]{75\frac{1}{8}}}{\sqrt[3]{180}} = 8 \sqrt[3]{\frac{1,215}{2,880}} = 8 \sqrt[3]{\frac{27}{64}} =$$

$8 \times \frac{3}{4} = 6$ inches. The process, however, is longer and is not so direct, and the first method is to be preferred.

326. If two cylinders have *equal* volumes, but different diameters, the diameters are to each other inversely as the square roots of their lengths. Hence, if it is desired to find the diameter of a cylinder that is to be 15 inches long, and which shall have the same volume as one that is 9 inches in diameter and 12 inches long, we write the proportion

$$9 : x = \sqrt{15} : \sqrt{12}.$$

Since neither 12 nor 15 are perfect squares, we square all of the terms (Arts. 325 and 314) and obtain

$$81 : x^2 = 15 : 12; \text{ whence } x^2 = \frac{81 \times 12}{15} = 64.8,$$

and $x = \sqrt{64.8} = 8.05$ inches = diameter of 15-inch cylinder.

EXAMPLES FOR PRACTICE.

327. Solve the following examples:

1. The intensity of light varies inversely as the square of the distance from the source of light. If a gas jet illuminates an object 30 feet away with a certain distinctness, how much brighter will the object be at a distance of 20 feet? Ans. $2\frac{1}{4}$ times as bright.

2. In the last example, suppose that the object had been 40 feet from the gas jet; how bright would it have been compared with its brightness at 30 feet from the gas jet? Ans. $\frac{9}{16}$ as bright.

3. When comparing one light with another, the intensities of their illuminating powers vary as the squares of their distances from the

source. If a man can just distinguish the time indicated by his watch, 50 feet from a certain light, at what distance could he distinguish the time from a light 3 times as powerful? Ans. 86.6+ feet.

4. The quantity of air flowing through a mine varies directly as the square root of the pressure. If 60,000 cubic feet of air flow per minute when the pressure is 2.8 pounds per square foot, how much will flow when the pressure is 3.6 pounds per square foot?

Ans. 68,034 cu. ft. per min., nearly.

5. In the last example, suppose that 70,000 cubic feet per minute had been required; what would be the pressure necessary for this quantity? Ans. 3.81+ lb. per sq. ft.

CAUSES AND EFFECTS.

328. Many examples in proportion may be more easily solved by using the principle of *cause and effect*. That which may be regarded as producing a change or alteration in something, or as accomplishing something, may be called a **cause**, and the change, or alteration, or thing accomplished, is the **effect**.

329. *Like causes produce like effects.* Hence, when two causes of the same kind produce two effects of the same kind, the ratio of the causes equals the ratio of the effects; in other words, the first cause is to the second cause as the first effect is to the second effect. Thus, in the question, if 3 men can lift 1,400 pounds, how many pounds can 7 men lift? we call 3 men and 7 men the *causes* (since they accomplish something, viz., the lifting of the weight), the number of pounds lifted, viz., 1,400 pounds and x pounds, are the effects. If we call 3 men the first cause, 1,400 pounds is the first effect; 7 men is the second cause, and x pounds is the second effect. Hence, we may write

$$\begin{array}{ccccccc} 1st\ cause & 2d\ cause & & 1st\ effect & 2d\ effect & & \\ 3 & : & 7 & = & 1,400 & : & x, \end{array}$$

whence $x = \frac{7 \times 1,400}{3} = 3,266\frac{2}{3}$ pounds.

330. The principle of cause and effect is extremely useful in the solution of examples in compound proportion, as we shall now show.

COMPOUND PROPORTION.

331. All the cases of proportion so far considered have been cases of **simple proportion**; i. e., each term has been composed of but one number. There are many cases, however, in which two or all of the terms have more than one number in them; all such cases belong to **compound proportion**. In all examples in compound proportion, both causes or both effects or all four consist of more than two numbers. We will illustrate this by an

EXAMPLE.—If 40 men earn \$1,280 in 16 days, how much will 36 men earn in 31 days?

SOLUTION.—Since 40 men earn something, 40 men is a cause, and since they take 16 days in which to earn something, 16 days is also a cause. For the same reason, 36 men and 31 days are also causes. The effects, that which is earned, are 1,280 dollars and x dollars. Then, 40 men and 16 days make up the first cause, and 36 men and 31 days make up the second cause. \$1,280 is the first effect and \$ x is the second effect. Hence, we write

$$\begin{array}{ccccccc} 1st\ cause & 2d\ cause & & 1st\ effect & 2d\ effect \\ 40 & : & 36 & = & 1,280 & : & x \\ 16 & & 31 & & & & \end{array}$$

Now, instead of using the colon to express the ratio, we shall use the vertical line (see Art. 289), and the above becomes

$$\begin{array}{c|c} 40 & 36 \\ 16 & 31 \end{array} = 1,280 \quad | \quad x.$$

In the last expression, the product of all of the numbers included between the vertical lines must equal the product of all the numbers without them; i. e., $36 \times 31 \times 1,280 = 40 \times 16 \times x$.

$$\text{Or } x = \frac{36 \times 31 \times 1,280}{40 \times 16} = \$2,282. \text{ Ans.}$$

332. The above might have been solved by canceling factors of the numbers in the original proportion. For if any number within the lines has a factor common to any number without the lines, that factor may be canceled from both numbers. Thus, 16 is contained in

$$\begin{array}{c|c} 40 & 36 \\ 16 & 31 \end{array} = \frac{2}{80} \quad | \quad x,$$

1,280, 80 times. Cancel 16 and 1,280, and write 80 above 1,280. 40 is contained in 80, 2 times. Cancel 40 and 80,

and write 2 above 80. Now, since there are no more numbers that can be canceled, $x = 36 \times 31 \times 2 = \$2,232$, the same result as was obtained in the last article.

333. Rule 43.—Write all the numbers forming the first cause in a vertical column, and draw a vertical line; on the other side of this line write in a vertical column all of the numbers forming the second cause. Write the sign of equality to the right of the second column, and on the right of this form a third column of the numbers composing the first effect, drawing a vertical line to the right; on the other side of this line, write, for a fourth column, the numbers composing the second effect. There must be as many numbers in the second cause as in the first cause, and in the second effect as in the first effect; hence, if any term is wanting, write x in its place. Multiply together all of the numbers within the vertical lines, and also all those without the lines (canceling previously, if possible), and divide the product of those numbers which do not contain x by the product of the others in which x occurs, and the result will be the value of x .

334. EXAMPLE.—If 40 men can dig a ditch 720 feet long, 5 feet wide and 4 feet deep in a certain time, how long a ditch 6 feet deep and 3 feet wide could 24 men dig in the same time?

SOLUTION.—Here 40 men and 24 men are the causes and the two ditches are the effects. Hence,

$$\begin{array}{c|c|c|c}
 40 & 24 & = & x \\
 \frac{720}{5} & \frac{6}{4} & & \frac{3}{5}
 \end{array}
 \quad \text{whence, } x = 24 \times 5 \times 4 = 480 \text{ feet. Ans.}$$

335. EXAMPLE.—The volume of a cylinder varies directly as its length and directly as the square of its diameter. If the volume of a cylinder 10 inches in diameter and 20 inches long is 1,570.8 cubic inches, what is the volume of another cylinder 16 inches in diameter and 24 inches long?

SOLUTION.—In this example, either the dimensions or the volumes may be considered the causes; say we take the dimensions for the causes. Then, squaring the diameters,

$$\begin{array}{c|c|c|c|c|c}
 10^2 & 16^2 & = & 1,570.8 & x, \text{ or } & 100 & 256 & = & 1,570.8 & x; \\
 20 & 24 & & & & 20 & 24 & & & \\
 & & & & & 5 & 6 & & &
 \end{array}$$

whence, $x = \frac{256 \times 6 \times 1,570.8}{5 \times 100} = 4,825.4976$ cubic inches. Ans.

336. EXAMPLE.—If a block of granite 8 ft. long, 5 ft. wide and 3 ft. thick weighs 7,200 lb., what will be the weight of a block of granite 12 ft. long, 8 ft. wide and 5 ft. thick?

SOLUTION.—Taking the weights as the effects, we have

$$\begin{array}{c|c} \begin{array}{c} 8 \\ 5 \\ 3 \end{array} & \begin{array}{c} 4 \\ 12 \\ 7,200 \end{array} \end{array} \quad \left| \quad x, \text{ or } x = 4 \times 7,200 = 28,800 \text{ pounds.} \quad \text{Ans.} \right.$$

337. EXAMPLE.—If 12 compositors in 30 days of 10 hours each set up 25 sheets of 16 pages each, 32 lines to the page, in how many days 8 hours long can 18 compositors set up, in the same type, 64 sheets of 12 pages each, 40 lines to the page?

SOLUTION.—Here compositors, days, and hours compose the causes, and sheets, pages, and lines the effects. Hence,

$$\begin{array}{c|c} \begin{array}{c} 3 \\ 12 \\ 30 \\ 8 \\ 10 \end{array} & \begin{array}{c} 3 \\ 18 \\ x \\ 8 \\ 32 \end{array} \end{array} \quad \left| \quad \begin{array}{c} 2 \\ 25 \\ 16 \\ 32 \\ 40 \end{array} \right. \quad \left| \quad \begin{array}{c} 2 \\ 64 \\ 12 \\ 40 \\ 32 \end{array} \right. \quad \left| \quad \begin{array}{c} 2 \\ 12, \text{ or } x = 3 \times 10 \times 2 = 60 \text{ days.} \end{array} \quad \text{Ans.} \right.$$

338. In examples stated like that in Art. 335, should an inverse proportion occur, write the various numbers as in the preceding examples, and then transpose those numbers which are said to vary inversely from one side of the vertical line to the other side.

EXAMPLE.—The centrifugal force of a revolving body varies directly as its weight, as the square of its velocity and inversely as the radius of the circle described by the center of the body. If the centrifugal force of a body weighing 15 pounds is 187 pounds when the body revolves in a circle having a radius of 12 inches, with a velocity of 20 feet per second, what will be the centrifugal force of the same body when the radius is increased to 18 inches and the speed is increased to 24 feet per second?

SOLUTION.—Calling the centrifugal force the effect, we have,

$$\begin{array}{c|c} \begin{array}{c} 15 \\ 20^2 \\ 12 \end{array} & \begin{array}{c} 15 \\ 24^2 \\ 18 \end{array} \end{array} \quad \left| \quad \begin{array}{c} 187 \\ x \end{array} \right.$$

Transposing 12 and 18 (since the radii are to vary inversely) and squaring 20 and 24,

$$\begin{array}{c|c} \begin{array}{c} 15 \\ 25 \\ 400 \\ 18 \end{array} & \begin{array}{c} 15 \\ 2 \\ 36 \\ 576 \\ 12 \end{array} \end{array} \quad \left| \quad \begin{array}{c} 187 \\ x \end{array} \right. \quad \left| \quad x, \text{ or } x = \frac{12 \times 2 \times 187}{25} = 179.52 \text{ pounds.} \quad \text{Ans.} \right.$$

EXAMPLES FOR PRACTICE.

339. Solve the following by compound proportion:

1. If 12 men dig a trench 40 rods long in 24 days of 10 hours each, how many rods can 16 men dig in 18 days of 9 hours each?

Ans. 36 rods.

2. If a piece of iron 7 ft. long, 4 in. wide, and 6 in. thick weighs 600 lb., how much will a piece of iron weigh that is 16 ft. long, 8 in. wide and 4 in. thick?

Ans. 1,828 $\frac{1}{2}$ lb.

3. If 24 men can build a wall 72 rods long, 6 feet wide, and 5 feet high in 60 days of 10 hours each, how many days will it take 32 men to build a wall 96 rods long, 4 feet wide and 8 feet high, working 8 hours a day?

Ans. 80 days.

4. The horsepower of an engine varies as the mean effective pressure, as the piston speed and as the square of the diameter of the cylinder. If an engine having a cylinder 14 inches in diameter develops 112 horsepower when the mean effective pressure is 48 pounds per square inch and the piston speed is 500 feet per minute, what horsepower will another engine develop if the cylinder is 16 inches in diameter, piston speed is 600 feet per minute, and mean effective pressure is 56 pounds per square inch?

Ans. 204.8 horsepower.

5. Referring to the example in Art. 335, what will be the volume of a cylinder 20 inches in diameter and 24 inches long?

Ans. 7,539.84 cubic inches.

6. Knowing that the product of $3 \times 5 \times 7 \times 9$ is 945, what is the product of $6 \times 15 \times 14 \times 36$?

Ans. 45,360.

7. The speed in miles per hour of a locomotive is directly proportional to the diameter of its driving wheels and the number of revolutions they make in one minute. A locomotive with driving wheels 66 inches in diameter runs 29.45 miles in an hour when the wheels make 150 revolutions per minute; how many miles will be run in one hour by a locomotive having wheels 72 inches in diameter running 220 revolutions per minute?

Ans. 47.12 miles per hour.

8. The capacity of a cylindrical boiler is proportional to its length and the square of its diameter. A boiler 12 feet long and 48 inches in diameter will hold 1,128 gallons; what is the capacity of a boiler 16 feet long and 42 inches in diameter?

Ans. 1,151.5 gallons.

9. The power that may be transmitted by a belt is proportional to its width and the diameter and number of revolutions made by the pulley on which it runs. If a belt 12 inches wide will transmit 10 horsepower when running over a pulley 20 inches in diameter that makes 125 revolutions per minute, how many horsepower may be transmitted by a belt 8 inches wide when running over a pulley 30 inches in diameter that makes 200 revolutions per minute?

Ans. 16 horsepower.

10. The load that a beam supported at the two ends will carry is directly proportional to its width and the square of its depth, and inversely proportional to its length. If an oak beam 8 inches wide, 12 inches deep, and 15 feet long will safely carry a load of 13,824 pounds, what is the safe load that a beam 10 inches wide, 16 inches deep, and 20 feet long will support?

Ans. 23,040 pounds.

MENSURATION AND USE OF LETTERS IN ALGEBRAIC FORMULAS.

FORMULAS.

340. The term **formula**, as used in mathematics and in technical books, may be defined as *a rule in which symbols are used instead of words*; in fact, a formula may be regarded as a shorthand method of expressing a rule. Any formula can be expressed in words, and when so expressed it becomes a rule.

Formulas are much more convenient than rules; they show at a glance all the operations that are to be performed; they do not require to be read three or four times, as is the case with most rules, to enable one to understand their meaning; they take up much less space, both in the printed book and in one's note-book, than rules; in short, whenever a rule can be expressed as a formula, the formula is to be preferred.

As the term "quantity" is a very convenient one to use, we will define it. In mathematics, the word **quantity** is applied to anything that it is desired to subject to the ordinary operations of addition, subtraction, multiplication, etc., when we do not wish to be more specific and state exactly what the thing is. Thus, we can say "two or more numbers," or "two or more quantities;" the word quantity is more general in its meaning than the word number.

341. The signs used in formulas are the ordinary signs indicative of operations, and the signs of aggregation. All these signs are explained in arithmetic, but some of them will here be explained in order to refresh the student's memory.

The signs indicative of operations are six in number, viz.: $+$, $-$, $\times$, $\div$, $|$, $\sqrt{}$.

Division is indicated by the sign $\div$, or by placing a straight line between the two quantities. Thus, $25 \div 17$, $25 / 17$, and $\frac{25}{17}$ all indicate that 25 is to be divided by 17. When both quantities are placed on the same horizontal line, the straight line indicates that the quantity on the left is to be divided by that on the right. When one quantity is below the other, the straight line between indicates that the quantity above the line is to be divided by the one below it.

The sign ($\sqrt{}$) indicates that some root of the quantity to the right is to be taken; it is called the **radical sign**. To indicate what root is to be taken, a small figure, called the **index**, is placed within the sign, this being always omitted when the square root is to be indicated. Thus, $\sqrt{25}$ indicates that the square root of 25 is to be taken; $\sqrt[3]{25}$ indicates that the cube root of 25 is to be taken; etc.

The signs of aggregation are four in number; viz., --- , $()$, $[\]$, and $\{ \}$, respectively called the **vinculum**, the **parenthesis**, the **brackets**, and the **brace**; they are used when it is desired to indicate that all the quantities included by them are to be subjected to the same operation. Thus, if we desire to indicate that the sum of 5 and 8 is to be multiplied by 7, and we do not wish to actually add 5 and 8 before indicating the multiplication, we may employ any one of the four signs of aggregation as here shown: $\overline{5+8} \times 7$, $(5+8) \times 7$, $[5+8] \times 7$, $\{5+8\} \times 7$. The vinculum is placed above those quantities which are to be treated as one quantity and subjected to the same operations.

While any one of the four signs may be used as shown above, custom has restricted their use somewhat. The vinculum is rarely used except in connection with the radical sign. Thus, instead of writing $\sqrt[3]{(5+8)}$, $\sqrt[3]{[5+8]}$, or $\sqrt[3]{\{5+8\}}$ for the cube root of 5 plus 8, all of which would be correct, the vinculum is nearly always used, $\sqrt[3]{\overline{5+8}}$.

In cases where but one sign of aggregation is needed (except, of course, when a root is to be indicated), the parenthesis

is always used. Hence, $(5 + 8) \times 7$ would be the usual way of expressing the product of 5 plus 8, and 7.

If two signs of aggregation are needed, the brackets and parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20 - 5) \div 3] \times 9$ means that the difference between 20 and 5 is to be divided by 3, and this result multiplied by 9.

If three signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20 - 5) \div 3] \times 9 - 21\} \div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

Should it be necessary to use all four of the signs of aggregation, the brace would be put outside, the brackets next, the parenthesis next, and the vinculum inside. For example, $\{[(\overline{20 - 5} \div 3) \times 9 - 21] \div 8\} \times 12$.

As stated in arithmetic, when several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of multiplication must always be performed first. Thus, $2 + 3 \times 4$ equals 14, 3 being multiplied by 4, before adding to 2. Similarly, $10 \div 2 \times 5$ equals 1, since 2×5 equals 10, and $10 \div 10$ equals 1. Hence, in the above case, if the brace were omitted, the result would be $\frac{1}{4}$, whereas, by inserting the brace, the result is 36.

Following the sign of multiplication comes the sign of division in order of importance. For example, $5 - 9 \div 3$ equals 2, 9 being divided by 3 before subtracting from 5. The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs, the indicated operations may be performed in the order in which the quantities are placed.

There is one other sign used, which is neither a sign of aggregation nor a sign indicative of an operation to be

performed; it is ($=$), and is called the sign of *equality*; it means that all on one side of it is exactly equal to all on the other side. For example, $2 = 2$, $5 - 3 = 2$, $5 \times (14 - 9) = 25$.

342. Having called particular attention to certain signs used in formulas, the formulas themselves will now be explained. First, consider the well-known rule for finding the horsepower of a steam-engine, which may be stated as follows:

Divide the continued product of the mean effective pressure in pounds per square inch, the length of the stroke in feet, the area of the piston in square inches, and the number of strokes per minute, by 33,000; the result will be the horsepower.

This is a very simple rule, and very little, if anything, will be saved by expressing it as a formula, so far as clearness is concerned. The formula, however, will occupy a great deal less space, as we shall show.

An examination of the rule will show that four quantities (viz., the mean effective pressure, the length of the stroke, the area of the piston, and the number of strokes) are multiplied together, and the result is divided by 33,000. Hence, the rule might be expressed as follows:

$$\text{Horsepower} = \frac{\text{mean effective pressure}}{(\text{in pounds per square inch})} \times \frac{\text{stroke}}{(\text{in feet})} \times \frac{\text{area of piston}}{(\text{in square inches})} \times \frac{\text{number of strokes}}{(\text{per minute})} \div 33,000.$$

This expression could be shortened by representing each quantity by a single letter; thus, representing horsepower by the letter " H ," the mean effective pressure in pounds per square inch by " P ," the length of stroke in feet by " L ," the area of the piston in square inches by " A ," the number of strokes per minute by " N ," and substituting these letters for the quantities that they represent, the above expression would reduce to

$$H = \frac{P \times L \times A \times N}{33,000},$$

a much simpler and shorter expression. This last expression is called a *formula*.

The formula just given shows, as we stated in the beginning, that a formula is really a shorthand method of expressing a rule. It is customary, however, to omit the sign of multiplication between two or more quantities when they are to be multiplied together, or between a number and a letter representing a quantity, it being always understood that, when two letters are adjacent with no sign between them, the quantities represented by these letters are to be multiplied. Bearing this fact in mind, the formula just given can be further simplified to

$$H = \frac{PLAN}{33,000}.$$

The sign of multiplication, evidently, can not be omitted between two or more numbers, as it would then be impossible to distinguish the numbers. A near approach to this, however, may be attained by placing a dot between the numbers which are to be multiplied together, and this is frequently done in works on mathematics when it is desired to economize space. In such cases it is usual to put the dot higher than the position occupied by the decimal point. Thus, $2 \cdot 3$ means the same as 2×3 ; $542 \cdot 749 \cdot 1,006$ indicates that the numbers 542, 749, and 1,006 are to be multiplied together.

It is also customary to omit the sign of multiplication in expressions similar to the following: $a \times \sqrt{b+c}$, $3 \times (b+c)$, $(b+c) \times a$, etc., writing them $a\sqrt{b+c}$, $3(b+c)$, $(b+c)a$, etc. The sign is not omitted when several quantities are included by a vinculum, and it is desired to indicate that the quantities so included are to be multiplied by another quantity. For example, $3 \times \overline{b+c}$, $\overline{b+c} \times a$, $\sqrt{b+c} \times a$, etc., are always written as here printed.

343. Before proceeding further, we will explain one other device that is used by formula makers and which is apt to puzzle one who encounters it for the first time—it is the use of what mathematicians call *primes* and *subs.*, and what printers call *superior* and *inferior* characters. As a rule, formula makers designate quantities by the initial

letters of the names of the quantities. For example, they represent volume by v , pressure by p , height by h , etc. This practice is to be commended, as the letter itself serves in many cases to identify the quantity which it represents. Some authors carry the practice a little further, and represent all quantities of the same nature by the same letter throughout the book, always having the same letter represent the same thing. Now, this practice necessitates the use of the primes and subs. above mentioned, when two quantities have the same name but represent different things. Thus, consider the word *pressure* as applied to steam, at different stages between the boiler and the condenser. First, there is *absolute* pressure, which is equal to the gauge pressure in pounds per square inch plus the pressure indicated by the barometer reading (usually assumed in practice to be 14.7 pounds per square inch, when a barometer is not at hand). If this be represented by p , how shall we represent the gauge pressure? Since the absolute pressure is always greater than the gauge pressure, suppose we decide to represent it by a capital letter, and the gauge pressure by a small (lower-case) letter. Doing so, P represents absolute pressure, and p , gauge pressure. Further, there is usually a "drop" in pressure between the boiler and the engine, so that the initial pressure, or pressure at the beginning of the stroke, is less than the pressure at the boiler. How shall we represent the initial pressure? We may do this in one of three ways and still retain the letter p or P to represent the word pressure: First, by the use of the prime mark; thus, p' or P' (read p *prime* and P *major prime*) may be considered to represent the initial gauge pressure, or the initial absolute pressure. Second, by the use of sub. figures; thus, p_1 or P_1 (read p *sub. one* and P *major sub. one*). Third, by the use of sub. letters; thus, p_i or P_i (read p *sub. i* and P *major sub. i*). In the same manner p'' (read p *second*), $p_{..}$, or p_r might be used to represent the gauge pressure at release, etc. The sub. letters have the advantage of still further identifying the quantity represented; in many instances, however, it is not convenient to use them, in which case

primes and subs. are used instead. The prime notation may be continued as follows: p''' , p^{iv} , p^v , etc.; it is inadvisable to use superior figures, for example, p^1 , p^2 , p^3 , p^a , etc., as they are liable to be mistaken for exponents.

The main thing to be remembered by the student is that *when a formula is given in which the same letters occur several times, all like letters having the same primes or subs. represent the same quantities, while those which differ in any respect represent different quantities.* Thus, in the formula

$$t = \frac{w_1 s_1 t_1 + w_2 s_2 t_2 + w_3 s_3 t_3}{w_1 s_1 + w_2 s_2 + w_3 s_3},$$

w_1 , w_2 , and w_3 represent the weights of three different bodies; s_1 , s_2 , and s_3 , their specific heats; and t_1 , t_2 , and t_3 , their temperatures; while t represents the final temperature after the bodies have been mixed together. It should be noted that those letters having the *same* subs. refer to the same bodies. Thus, w_1 , s_1 , and t_1 all refer to one of the three bodies; w_2 , s_2 , t_2 , to another body; etc.

It is very easy to apply the above formula when the values of the quantities represented by the different letters are known. All that is required is to substitute the numerical values of the letters, and then perform the indicated operations. Thus, suppose that the values of w_1 , s_1 , and t_1 are, respectively, 2 pounds, .0951, and 80° ; of w_2 , s_2 , and t_2 , 7.8 pounds, 1, and 80° ; and of w_3 , s_3 , and t_3 , $3\frac{1}{4}$ pounds, .1138, and 780° ; then, the final temperature t is, substituting these values for their respective letters in the formula,

$$t = \frac{2 \times .0951 \times 80 + 7.8 \times 1 \times 80 + 3\frac{1}{4} \times .1138 \times 780}{2 \times .0951 + 7.8 \times 1 + 3\frac{1}{4} \times .1138} = \frac{15.216 + 624 + 288.483}{.1902 + 7.8 + .36985} = \frac{927.699}{8.36005} = 110.97^\circ.$$

In substituting the numerical values, the signs of multiplication are, of course, written in their proper places; all the multiplications are performed before adding, according to the rule previously given.

344. The student should now be able to apply any formula involving only algebraic expressions that he may meet with, and which does not require the use of logarithms for its solution. We will, however, call his attention to one or two other facts that he may have forgotten.

Expressions similar to $\frac{\frac{160}{660}}{25}$ sometimes occur, the heavy line

indicating that 160 is to be divided by the quotient obtained by dividing 660 by 25. If both lines were light it would be impossible to tell whether 160 was to be divided by $\frac{660}{25}$, or whether $\frac{160}{660}$ was to be divided by 25. If this latter

result were desired, the expression would be written $\frac{\frac{160}{660}}{25}$. In every case the heavy line indicates that all above it is to be divided by all below it.

In an expression like the following, $\frac{160}{7 + \frac{660}{25}}$, the heavy

line is not necessary, since it is impossible to mistake the operation that is required to be performed. But, since $7 + \frac{660}{25} = \frac{175 + 660}{25}$, if we substitute $\frac{175 + 660}{25}$ for $7 + \frac{660}{25}$, the heavy line becomes necessary in order to make the resulting expression clear. Thus,

$$\frac{160}{7 + \frac{660}{25}} = \frac{160}{\frac{175 + 660}{25}} = \frac{160}{\frac{835}{25}}$$

Fractional exponents are sometimes used instead of the radical sign. That is, instead of indicating the square, cube, fourth root, etc., of some quantity, as 37, by $\sqrt{37}$, $\sqrt[3]{37}$, $\sqrt[4]{37}$, etc., these roots are indicated by $37^{\frac{1}{2}}$, $37^{\frac{1}{3}}$, $37^{\frac{1}{4}}$, etc. Should the numerator of the fractional exponent be some quantity other than 1, this quantity, whatever it may be, indicates that the quantity affected by the exponent is to be raised to the power indicated by the numerator; the

denominator is *always* the index of the root. Hence, instead of writing $\sqrt[4]{37^2}$ for the cube root of the square of 37, it may be written $37^{\frac{2}{3}}$, the denominator being the index of the root; in other words, $\sqrt[4]{37^2} = 37^{\frac{2}{3}}$. Likewise, $\sqrt[4]{(1+a^2b)^2}$ may also be written $(1+a^2b)^{\frac{2}{3}}$, a much simpler expression.

345. We will now give several examples showing how to apply some of the more difficult formulas that the student may encounter.

1. The area of any segment of a circle that is less than (or equal to) a semicircle is expressed by the formula

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h),$$

in which A = area of segment;

$\pi = 3.1416$;

r = radius;

E = angle obtained by drawing lines from the center to the extremities of arc of segment;

c = chord of segment;

and h = height of segment.

EXAMPLE.—What is the area of a segment whose chord is 10 inches long, angle subtended by chord is 83.46° , radius is 7.5 inches, and height of segment is 1.91 inches?

SOLUTION.—Applying the formula just given,

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h) = \frac{3.1416 \times 7.5^2 \times 83.46}{360} - \frac{10}{2}(7.5 - 1.91) = 40.968 - 27.95 = 13.018 \text{ square inches, nearly. Ans.}$$

2. The area of any triangle may be found by means of the following formula, in which A = the area, and a , b , and c represent the lengths of the sides:

$$A = \frac{b}{2} \sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b} \right)^2}.$$

EXAMPLE.—What is the area of a triangle whose sides are 21 feet, 46 feet, and 50 feet long?

SOLUTION.—In order to apply the formula, suppose we let a represent the side that is 21 feet long; b , the side that is 50 feet long; and c , the side that is 46 feet long. Then substituting in the formula,

$$\begin{aligned}
 A &= \frac{b}{2} \sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b} \right)^2} = \frac{50}{2} \sqrt{21^2 - \left(\frac{21^2 + 50^2 - 46^2}{2 \times 50} \right)^2} \\
 &= \frac{50}{2} \sqrt{441 - \left(\frac{441 + 2,500 - 2,116}{100} \right)^2} = 25 \sqrt{441 - \left(\frac{825}{100} \right)^2} \\
 &= 25 \sqrt{441 - 8.25^2} = 25 \sqrt{441 - 68.0625} = 25 \sqrt{372.9375} \\
 &= 25 \times 19.312 = 482.8 \text{ square feet, nearly. Ans.}
 \end{aligned}$$

The operations in the above examples have been extended much farther than was necessary; it was done in order to show the student every step of the process. The last formula is perfectly general, and the same answer would have been obtained had the 50-foot side been represented by a , the 46-foot side by b , and the 21-foot side by c .

3. The Rankine-Gordon formula for determining the least load in pounds that will cause a long column to break is

$$P = \frac{SA}{1 + q \frac{l^2}{G^2}},$$

in which P = load (pressure) in pounds;

S = ultimate strength (in pounds per square inch) of the material composing the column;

A = area of cross-section of column in square inches;

q = a factor (multiplier) whose value depends upon the shape of the ends of the column and on the material composing the column;

l = length of column in inches;

and G = least radius of gyration of cross-section of column.

The values of S , q , and G^2 are given in printed tables in books in which this formula occurs.

EXAMPLE.—What is the least load that will break a hollow wrought-iron column whose outside diameter is 14 inches; inside diameter, 11 inches; length, 20 feet, and whose ends are flat?

SOLUTION.—For steel, $S = 150,000$, and $q = \frac{1}{25,000}$ for flat-ended steel columns; A , the area of the cross-section, $= .7854(d_1^2 - d_2^2) = .7854(14^2 - 11^2)$, d_1 and d_2 being the outside and inside diameters,

respectively; $l = 20 \times 12 = 240$ inches; and $G^2 = \frac{d_1^2 + d_2^2}{16} = \frac{14^2 + 11^2}{16}$.
Substituting these values in the formula,

$$P = \frac{SA}{1 + \frac{l}{G^2}} = \frac{150,000 \times .7854 (14^2 - 11^2)}{1 + \frac{1}{25,000} \times \frac{240^2}{14^2 + 11^2}}$$

$$\frac{150,000 \times 58.905}{1 + .1163} = \frac{8,835,750}{1.1163} = 7,915,211 \text{ pounds. Ans.}$$

4. EXAMPLE.—When $A = 10$, $B = 8$, $C = 5$, and $D = 4$, what is the value of E in the following :

$$(a) E = \sqrt[3]{\frac{BCD}{A(2 + \frac{D^2}{C^2})}} \quad (b) E = \frac{A - \frac{1}{2}D + \frac{4}{15}B^2}{A - \frac{1}{2}D + \frac{2}{32}B^2}$$

SOLUTION.—(a) Substituting,

$$E = \sqrt[3]{\frac{8 \times 5 \times 4}{10(2 + \frac{4^2}{5^2})}}$$

To simplify the denominator, square the 4 and 5, add the resulting fraction to 2, and multiply by 10. Simplifying, we have

$$E = \sqrt[3]{\frac{160}{10(2 + \frac{16}{25})}} = \sqrt[3]{\frac{160}{10 \times \frac{66}{25}}} = \sqrt[3]{\frac{160}{660}} = \sqrt[3]{\frac{200}{33}}$$

Reducing the fraction to a decimal before extracting the cube root,

$$E = \sqrt[3]{6.0606} = 1.823. \text{ Ans.}$$

(b) Substituting,

$$E = \frac{10 - \frac{1}{2} \times 4 + \frac{4 \times 8^2}{10 + 5}}{10 - \frac{1}{2} \times 4 + \frac{2 \times 8^2}{10 + 25}} = \frac{10 - 2 + \frac{4 \times 64}{15}}{10 - 2 + \frac{2 \times 64}{32}}$$

$$\frac{7 + 17.066 +}{10 - 2 + 4} = \frac{24.066 +}{8} = 3.008 +. \text{ Ans.}$$

345.—In the preceding pages, the unknown quantity has always been represented by the single letter at the left of the sign of equality, while the letters at the right have represented known values from which the required values could be found. It is possible, however, to find the value of the quantity represented by any letter in a formula, if the values represented by all the others are known. For example,

let it be required to find how many strokes per minute an engine having a piston area of 78.54 sq. in. must make in order to develop 60 horsepower, if the mean effective pressure is 40 lb. per sq. in., and the length of stroke $1\frac{1}{4}$ ft. By substituting the given values in the formula $H = \frac{PLAN}{33,000}$, we have

$$60 = \frac{40 \times 1\frac{1}{4} \times 78.54 \times N}{33,000},$$

in which N , the number of strokes, is to be found.

But it is evident that the expression on the right of the sign of equality is equal to $\frac{40 \times 1\frac{1}{4} \times 78.54}{33,000} \times N$, a fraction whose numerator is composed of 3 factors. Reducing the numerator to a single number by performing the indicated multiplications, we obtain, after canceling,

$$60 = \frac{119}{1,000} \times N = .119 N.$$

If 60 equals .119 N , then N equals 60 divided by .119; hence,

$$N = \frac{60}{.119} = 504.2 \text{ revolutions per minute.}$$

The method of procedure is essentially the same when the unknown quantity occurs in the denominator of a formula.

Thus, in the formula $f = \frac{mv^2}{r}$, suppose that $f = 375$, $m = 1.25$, and $v = 60$. Then, substituting,

$$375 = \frac{1.25 \times 60^2}{r} = \frac{4,500}{r}.$$

But if 375 equals 4,500 divided by r , then $375 \times r = 4,500$; hence, r must equal 4,500 divided by 375, or $r = \frac{4,500}{375} = 12$.

EXAMPLES FOR PRACTICE.

Find the numerical values of x in the following formulas, when $A = 9$, $B = 8$, $d = 10$, $e = 3$, and $c = 2$:

$$1. \quad x = \frac{d + c^2}{d^2 - 40}. \quad \text{Ans. } x = \frac{7}{80}.$$

$$2. \quad x = \frac{\frac{2}{3}(A + e)}{ce}. \quad \text{Ans. } x = 1\frac{1}{3}.$$

3. $x = \sqrt{\frac{d^2}{2c}} + \sqrt{AB^2}$. Ans. $x = 29$.
4. $x = \frac{Ae}{\sqrt{16Bc}} + \frac{5}{16}$. Ans. $x = 2$.
5. $x = (c + 2e)\left(\sqrt[3]{B} - \frac{1}{c}\right) + \frac{e^2 - c^2}{e^2 + c^2}$. Ans. $x = 12\frac{5}{13}$.
6. $x = \sqrt{\frac{Bcd}{A\left(2 + \frac{d^2}{e^2}\right)}}$. Ans. $x = .396 +$.

MENSURATION.

346. **Mensuration** treats of the measurement of lines, angles, surfaces, and solids.

LINES AND ANGLES.

347. A straight line is one that does not change its direction throughout its whole length. To distinguish one straight line from another, its two extreme points are designated by letters. The line shown in Fig. 1 would be called the line AB .

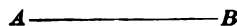


FIG. 1.

348. A curved line changes its direction at every point. Curved lines are designated by three or more letters, as the curved line ABC , Fig. 2.

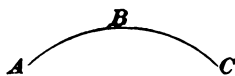


FIG. 2.

349. **Parallel lines** (Fig. 3) are those which are equally distant from each other at all points.

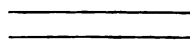


FIG. 3.

350. A line is **perpendicular** to another (see Fig. 4) when it meets that line so as not to incline towards it on either side.



FIG. 4.

351. A **vertical line** is one that points towards the center of the earth, and is also known as a *plumb* line.

352. A **horizontal line** (see Fig. 5) is one that makes a right angle with any vertical line.

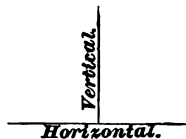


FIG. 5.

353. An **angle** is the opening between two lines which intersect or meet; the point of meeting is called the **vertex** of the angle. Angles are distinguished by naming the vertex and a point on each line. Thus, in Fig. 6, the angle formed by the lines AB and CB is called the angle ABC , or the angle CBA ; the letter at the vertex is always placed in the middle. When an angle stands alone so that it can not be mistaken for any other angle, only the vertex letter need be used. Thus, the angle referred to might be designated simply as the angle B .

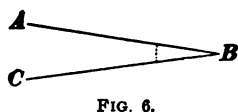


FIG. 6.

354. If one straight line meets another straight line at a point between its ends, as in Fig. 7, two angles, ABC and ABD , are formed, which are called **adjacent angles**.

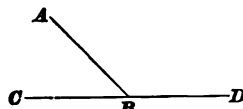


FIG. 7.

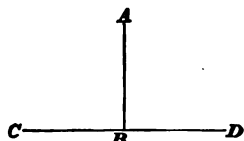


FIG. 8.

355. When these adjacent angles, ABC and ABD , are equal, as in Fig. 8, they are called **right angles**.

356. An **acute angle** is less than a right angle. ABC , Fig. 9, is an acute angle.

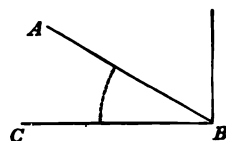


FIG. 9.

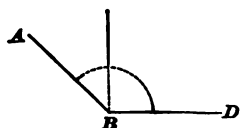


FIG. 10.

357. An **obtuse angle** is greater than a right angle. ABD (Fig. 10) is an obtuse angle.

358. A **circle** (see Fig. 11) is a figure bounded by a curved line, called the **circumference**, every point of which is equally distant from a point within, called the **center**.

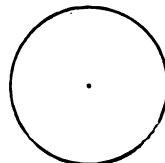


FIG. 11.

359. An **arc** of a circle is any part of its circumference; thus $a c b$, Fig. 12, is an arc of the circle.

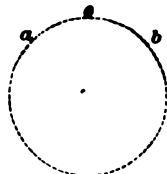


FIG. 12.

The circumference of every circle is considered to be divided into 360 equal parts, or arcs, called **degrees**; every degree is subdivided into 60 equal parts, called **minutes**, and every minute is again divided into 60 equal parts, called **seconds**.

Since 1 degree is $\frac{1}{360}$ of any circumference, it follows that the length of a degree will be different in circles of different sizes, but the proportion of the length of an arc of one degree to the whole circumference will always be the same, viz., $\frac{1}{360}$ of the circumference.

Degrees, minutes, and seconds are denoted by the symbols $^{\circ}$, $'$, $''$. Thus, the expression $37^{\circ} 14' 41''$ is read 37 degrees, 14 minutes, and 44 seconds.

360. The arcs of circles are used to measure angles.

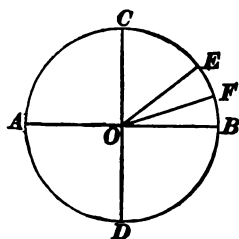


FIG. 13.

An angle having its vertex at the center of a circle is measured by the arc included between its sides; thus, in Fig. 13, the arc FB measures the angle FOB . If the arc FB contains 20° , or $\frac{20}{360}$ of the circumference, the angle FOB would be an angle of 20° ; if it contained $20^{\circ} 14' 18''$, it would be an angle of $20^{\circ} 14' 18''$, etc.

In the figure, if the line CD be drawn perpendicular to AB , the adjacent angles will be equal, and the circle will be divided into four equal angles, each of which will be a right angle. A right angle, therefore, is an angle of $\frac{360^{\circ}}{4}$, or 90° ; two right angles are measured by 180° , or half the circumference, and four right angles by the whole circumference, or 360° . One-half of a right angle, as EOB , is an

angle of 45° . An *acute* angle may now be defined as an angle of *less* than 90° , and an *obtuse* angle as one of *more* than 90° . These values are important, and should be remembered.

361. From the foregoing it will be evident that if a number of straight lines on the same side of a given straight line meet at the same point, the sum of all the angles formed is equal to two right angles, or 180° . Thus, in Fig. 14, angles $C O B + D O C + E O D + F O E + A O F = 2$ right angles, or 180° .

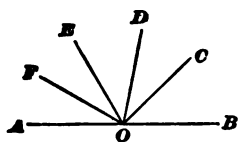


FIG. 14.

362. Also, if through a given point any number of straight lines be drawn, the sum of all the angles formed about the points of intersection equals four right angles, or 360° . Thus, in Fig. 15, angles $H O F + F O C + C O A + A O G + G O E + E O D + D O B + B O H =$ four right angles, or 360° .

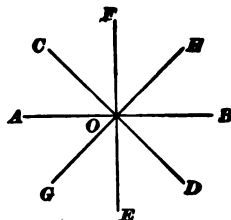


FIG. 15.

EXAMPLE.—In a fly-wheel, with 12 arms, how many degrees are there in the angle included between the center lines of any two arms, the arms being spaced equally?

SOLUTION.—Since there are 12 arms, there are 12 angles, which together equal 360° . Hence, one angle equals $\frac{1}{12}$ of 360° , or $\frac{360^\circ}{12} = 30^\circ$.

Ans.

EXAMPLES FOR PRACTICE.

- How many seconds in $32^\circ 14' 6''$? Ans. 116,046 sec.
- How many degrees, minutes, and seconds do 38,582 seconds amount to? Ans. $10^\circ 43' 2''$.
- How many right angles are there in an angle of 170° ? Ans. $1\frac{1}{2}$ right angles.
- In a pulley with five arms, what part of a right angle is included between the center lines of any two arms? Ans. $\frac{1}{4}$ of a right angle.
- If one straight line meets another so as to form an angle of $20^\circ 10'$, what does its adjacent angle equal? Ans. $159^\circ 50'$.
- If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, how many degrees are there in each angle? Ans. 30° .

QUADRILATERALS.

363. A **plane figure** is any part of a plane or flat surface bounded by straight or curved lines.

364. A **quadrilateral** is a plane figure bounded by four straight lines.

365. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

There are four kinds of parallelograms: the **square**, the **rectangle**, the **rhombus**, and the **rhomboid**.

366. A **rectangle** (Fig. 16) is a parallelogram whose angles are all right angles.

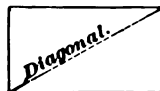


FIG. 16.

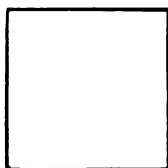


FIG. 17.

367. A **square** (Fig. 17) is a rectangle whose sides are all of the same length.

368. A **rhomboid** (Fig. 18) is a parallelogram whose opposite sides are equal and parallel, and whose angles are not right angles.

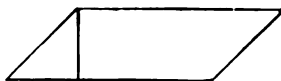


FIG. 18.

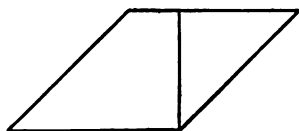


FIG. 19.

369. A **rhombus** (Fig. 19) is a parallelogram having equal sides, and whose angles are not right angles.

370. A **trapezoid** (Fig. 20) is a quadrilateral which has only two of its sides parallel.

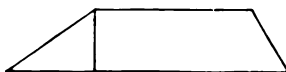


FIG. 20.

371. The **altitude** of a parallelogram is the perpendicular distance between the parallel sides, as shown by the dotted lines in Figs. 18, 19, and 20.

372. The **base** of any plane figure is the side on which it is supposed to stand.

373. The **area** of a surface is expressed by the number of unit squares it will contain.

374. A **unit square** is the square having a unit for its side. For example, if the unit is 1 inch, the unit square is the square each of whose sides measures 1 inch in length, and the area of a surface would be expressed by the number of square inches it would contain. If the unit were 1 foot, the unit square would measure 1 foot on each side, and the area of the given surface would be the number of square feet it would contain, etc.

The square that measures one inch on a side is called a **square inch**, and the one that measures one foot on a side is called a **square foot**. Square inch and square foot are abbreviated to sq. in. and sq. ft.

375. To find the area of any parallelogram:

Rule 44.—*Multiply the base by the altitude.*

NOTE.—Before multiplying, the base and altitude must be reduced to the same kind of units; that is, if the base should be given in feet and the altitude in inches, they could not be multiplied together until either the altitude had been reduced to feet, or the base to inches. This principle holds throughout the subject of mensuration.

EXAMPLE.—The sides of a square piece of sheet iron are each $10\frac{1}{2}$ inches long. How many square inches does it contain?

SOLUTION.— $10\frac{1}{2}$ inches = 10.25 inches when reduced to a decimal. The base and altitude are each 10.25 inches. Multiplying them together, $10.25 \times 10.25 = 105.06 +$ sq. in. Ans.

EXAMPLE.—What is the area in square rods of a piece of land in the shape of a rhomboid, one side of which is 8 rods long, and whose length, measured on a line perpendicular to this side, is 200 feet?

SOLUTION.—The base is 8 rods and the altitude 200 feet. As the answer is to be in rods, the 200 feet should be reduced to rods. Reducing $200 \div 16\frac{1}{2} = 200 \div \frac{33}{2} = 12.12$ rods. Hence, area = $8 \times 12.12 = 96.96$ sq. rd. Ans.

376. To find the area of a trapezoid:

Rule 45.—*Multiply one-half the sum of the parallel sides by the altitude.*

EXAMPLE.—A board 14 feet long is 20 inches wide at one end and 16 inches wide at the other. If the ends are parallel, how many square feet does the board contain?

SOLUTION.—One-half the sum of the parallel sides = $\frac{20 + 16}{2} = 18$ inches = $1\frac{1}{2}$ feet. The length of the board corresponds to the altitude of a trapezoid. Hence, $14 \times 1\frac{1}{2} = 21$ sq. ft. Ans.

377. Having given the area of a parallelogram and one dimension, to find the other dimension:

Rule 46.—*Divide the area by the given dimension.*

EXAMPLE.—What is the width of a parallelogram whose area is 212 square feet and whose length is $26\frac{1}{2}$ feet?

SOLUTION.— $212 \div 26\frac{1}{2} = 212 \div \frac{53}{2} = 8$ feet. Ans.

The following examples illustrate a few special cases:

EXAMPLE.—An engine room is 22 feet by 32 feet. The engine-bed occupies a space of 3 feet by 12 feet; the fly-wheel pit, a space of 2 feet by 6 feet, and the outer bearing, a space of 2 feet by 4 feet. How many square feet of flooring will be required for the room?

SOLUTION.—Area of engine-bed = $3 \times 12 = 36$ sq. ft.
 Area of fly-wheel pit = $2 \times 6 = 12$ sq. ft.
 Area of outer bearing = $2 \times 4 = 8$ sq. ft.
 Total, $\overline{56}$ sq. ft.
 Area of engine room = $22 \times 32 = 704$ sq. ft.
 $704 - 56 = 648$ square feet of flooring required. Ans.

EXAMPLE.—How many square yards of plaster will it take to cover the sides and ceiling of a room 16×20 feet and 11 feet high, having four windows, each 7×4 feet, and three doors, each 9×4 feet over all, the baseboard coming 6 inches above the floor?

SOLUTION.—

Area of ceiling = $16 \times 20 = 320$ sq. ft.
 Area of end walls = $2(16 \times 11) = 352$ sq. ft.
 Area of side walls = $2(20 \times 11) = 440$ sq. ft.
 Total area = $\overline{1,112}$ sq. ft.

From the above must be deducted:

Windows = $4(7 \times 4) = 112$ sq. ft.
 Doors = $3(9 \times 4) = 108$ sq. ft.

Baseboard less the width of three doors = $(72 - 12) \times \frac{6}{12} = 30$ sq. ft.

Total number of feet to be deducted = $112 + 108 + 30 = 250$ sq. ft.

Hence, number of square feet to be plastered = $1,112 - 250 = 862$ sq. ft., or $95\frac{1}{3}$ sq. yd. Ans.

EXAMPLE.—How many acres are contained in a rectangular tract of land 800 rods long and 520 rods wide?

SOLUTION.— $800 \times 520 = 416,000$ sq. rd. Since there are 160 square rods in one acre, the number of acres = $416,000 \div 160 = 2,600$ acres. Ans.

EXAMPLES FOR PRACTICE.

1. What is the area in square feet of a rhombus whose base is 84 inches, and whose altitude is 3 feet? Ans. 21 sq. ft.
2. A flat roof, 46 feet by 80 feet in size, is covered by tin roofing weighing one-half pound per square foot; what is the total weight of the roof? Ans. 1,840 lb.
3. One side of a room measures 16 ft. If the floor contains 240 square feet, what is the length of the other side? Ans. 15 ft.
4. How many square feet in a board 12 feet long, 18 inches wide at one end, and 12 inches wide at the other end? Ans. 15 sq. ft.
5. How much would it cost to lay a sidewalk a mile long and 8 feet 6 inches wide, at the rate of 20 cents per square foot? How much at the rate of \$1.80 per square yard? Ans. \$8,976 in each case.
6. How many square yards of plastering will be required for the ceiling and walls of a room 10×15 feet and 9 feet high; the room contains one door $3\frac{1}{2} \times 7$ feet, three windows $3\frac{1}{2} \times 6$ feet, and a baseboard 8 inches high? Ans. 53.5 sq. yd.

THE TRIANGLE.

378. A **triangle** is a plane figure having three sides.

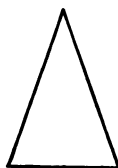


FIG. 21.

379. An **isosceles** triangle is one having two of its sides equal; see Fig. 21.

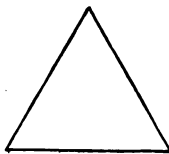


FIG. 22.

380. An **equilateral** triangle (Fig. 22) is one having all of its sides of the same length.



FIG. 23.

381. A **scalene** triangle (Fig. 23) is one having no two of its sides equal.

382. A **right-angled** triangle (Fig. 24) is any triangle having one right angle. The side opposite the right angle is called the **hypotenuse**.

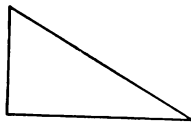


FIG. 24.

In any triangle the sum of the three angles equals two right angles, or 180° . Thus, in Fig. 25, the sum of the angles A , B , and C equals two right angles, or 180° . Hence, if any two angles of a triangle are given and it is required to find the third angle:

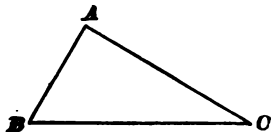


FIG. 25.

Rule 47.—*Add together the two given angles, and subtract their sum from 180° ; the result will be the third angle.*

EXAMPLE.—If two angles of a triangle = $48^\circ 16'$ and $47^\circ 50'$ respectively, what does the third angle equal?

SOLUTION.—First reduce $48^\circ 16'$ and $47^\circ 50'$ to minutes, for convenience in adding and subtracting the angles. $48^\circ = 48 \times 60 = 2,880'$; $2,880' + 16' = 2,896'$; hence, $48^\circ 16' = 2,896'$. In like manner, $47^\circ 50' = 47 \times 60 = 2,820' + 50' = 2,870'$. Adding the two angles together, and subtracting from 180° reduced to minutes, $2,896' + 2,870' = 5,766'$; $180^\circ = 180 \times 60 = 10,800'$; $10,800 - 5,766 = 5,034'$. Reducing this last number to degrees and minutes, $\frac{5,034}{60} = 83\frac{54}{60}^\circ = 83^\circ 54'$. Hence, the third angle in the triangle = $83^\circ 54'$. Ans.

383. In any right-angled triangle there can be but one right angle, and since the sum of all the angles is two right angles, it is evident that the sum of the two acute angles must equal one right angle, or 90° . Therefore, if in any right-angled triangle one acute angle is known, to find the other acute angle:

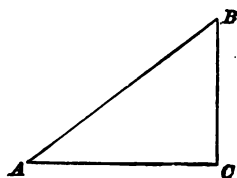


FIG. 26.

Rule 48.—*Subtract the known acute angle from 90° ; the result will be the other acute angle.*

EXAMPLE.—If one acute angle, as A , of the right-angled triangle ABC , Fig. 26, equals 30° , what does the angle B equal?

SOLUTION.— $90^\circ - 30^\circ = 60^\circ$. Ans.

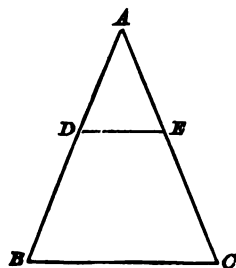


FIG. 27.

384. If a straight line be drawn through two sides of a triangle, parallel to the third side, a second triangle will be formed whose sides will be proportional to the corresponding sides of the first triangle. Thus, in the triangle ABC , Fig. 27, if the line DE be drawn parallel to the side BC , the triangle ADE will be formed and we shall have

- (1) Side AD : side DE = side AB : side BC ; and,
- (2) Side AE : side DE = side AC : side BC ; also,
- (3) Side AD : side AE = side AB : side AC .

EXAMPLE.—In Fig. 27, if $AB = 24$, $BC = 18$, and $DE = 8$, what does AD equal?

SOLUTION.—Writing these values for the sides in (1),

$$AD : 8 = 24 : 18; \text{ whence } AD = \frac{24 \times 8}{18} = 10\frac{2}{3}. \text{ Ans.}$$

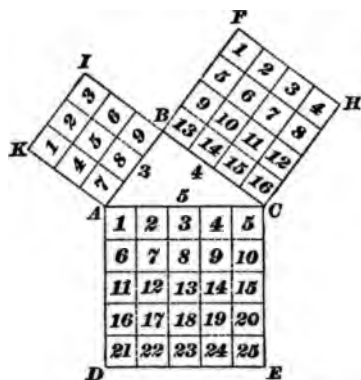


FIG. 28.

two sides forming the right angle in a right-angled triangle, to find the hypotenuse:

Rule 49.—*Square each of the sides forming the right angle; add the squares together, and take the square root of the sum.*

EXAMPLE.—If $AB = 3$ inches and $BC = 4$ inches, what is the length of the hypotenuse AC ?

SOLUTION.—Squaring each of the given sides, $3^2 = 9$ and $4^2 = 16$. Taking the square root of the sum of 9 and 16, the hypotenuse = $\sqrt{9 + 16} = \sqrt{25} = 5$ inches. Ans.

If the hypotenuse and one side are given, the other side can be found as follows:

Rule 50.—*Subtract the square of the given side from the square of the hypotenuse, and extract the square root of the remainder.*

EXAMPLE.—The side given is 3 inches, the hypotenuse is 5 inches; what is the length of the other side?

SOLUTION.— $3^2 = 9$; $5^2 = 25$. $25 - 9 = 16$, and the $\sqrt{16} = 4$ inches. Ans.

385. In any right-angled triangle, the square described on the hypotenuse is equal to the sum of the squares described upon the other two sides. If ABC , Fig. 28, is a right-angled triangle, right-angled at B , then the square described upon the hypotenuse AC is equal to the sum of the squares described upon the sides AB and BC .

Hence, having given the

EXAMPLE.—If from a church steeple which is 150 feet high, a rope is to be attached to the top, and to a stake in the ground, which is 85 feet from the center of the base (the ground being supposed to be level), what must be the length of the rope?

SOLUTION.—In Fig. 29, AB represents the steeple, 150 feet high; C a stake 85 feet from the foot of the steeple, and AC the rope. Here we have a right-angled triangle, right-angled at B , and AC is the hypotenuse. The square of $AB = 150^2 = 22,500$; of CB , $85^2 = 7,225$. $22,500 + 7,225 = 29,725$; $\sqrt{29,725} = 172.4$ feet, nearly. Ans.

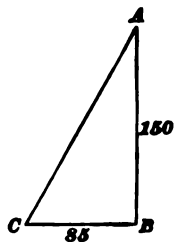


FIG. 29.

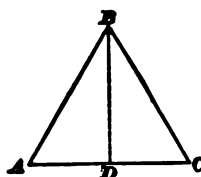


FIG. 30.

386. The altitude of any triangle is a line, as BD , drawn from the vertex B of the angle opposite the base AC , perpendicular to the base, as in Fig. 30, or

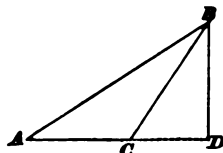


FIG. 31.

to the base extended, as in Fig. 31.

If in any parallelogram a straight line, called the **diagonal**, be drawn, connecting two opposite corners it will divide the parallelogram into two equal triangles, as ADB and DBC in Fig. 32. The area of each triangle will equal one-half the area of the parallelogram, or one-half the product of the base and the altitude. Hence, to find the area of any triangle:

Rule 51.—Multiply the base by the altitude, and divide the product by 2.

EXAMPLE.—What is the area in square feet of a triangle whose base is 18 feet, and whose altitude is 7 feet 9 inches?

SOLUTION.—7 feet 9 inches = $7\frac{3}{4}$ feet = $\frac{31}{4}$ feet. $18 \times \frac{31}{4} = 139\frac{1}{2}$, and one-half of $139\frac{1}{2} = 69\frac{1}{4}$ square feet. Ans.

To find the altitude or base of a triangle, having given the area and the base or altitude:

Rule 52.—Multiply the area by 2, and divide by the given dimension.

EXAMPLE.—What must be the height of a triangular piece of sheet metal to contain 100 square inches, if the base is 10 inches long?

SOLUTION.— $100 \times 2 = 200$; $200 \div 10 = 20$ inches. **Ans.**

EXAMPLES FOR PRACTICE.

1. What is the area of a triangle whose base is 18 feet long, and whose altitude is 10 feet 6 inches? **Ans.** 94.5 sq. ft.
2. Two angles of a scalene triangle together equal $100^\circ 4'$. What is the size of the third angle? **Ans.** $79^\circ 56'$.
3. One angle of a right-angled triangle equals $20^\circ 10' 5''$. What is the size of the other acute angle? **Ans.** $69^\circ 49' 55''$.
4. A ladder 65 feet long reaches to the top of a wall when its foot is 25 feet from the wall. How high is the wall? **Ans.** 60 ft.
5. Draw a triangle, and through two of its sides draw a line parallel to the base. Letter the different lines, and then, without referring to the text, write out the proportions existing between the sides of the two triangles.
6. A triangular piece of sheet metal weighs 24 pounds. If the base of the triangle is 4 feet and its height 6 feet, how much does the metal weigh per square foot? **Ans.** 2 lb.
7. The area of a triangle is 16 square inches. If the altitude is 4 inches, what does the base measure? **Ans.** 8 in.
8. Two sides of a right-angled triangle are 92 feet and 69 feet long. How long is the hypotenuse? **Ans.** 115 ft.

POLYGONS.

387. A **polygon** is a plane figure bounded by straight lines. The term is usually applied to a figure having more than four sides. The bounding lines are called the **sides**, and the sum of the lengths of all the sides is called the **perimeter** of the polygon.

388. A **regular polygon** is one in which all of the sides and all of the angles are equal.

389. A polygon of five sides is called a **pentagon**; one with six sides a **hexagon**; one of seven sides a **hep-**

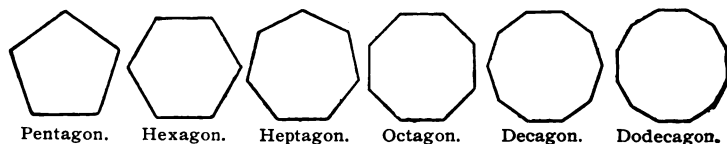


FIG. 33.

tagon, etc. Regular polygons having from five to twelve sides are shown in Fig. 33. In any polygon, the sum of all

the interior angles, as $A + B + C + D + E$, Fig. 34, equals 180° multiplied by a number which is two less than the number of sides in the polygon. Hence, to find the size of any one of the interior angles of a *regular* polygon:

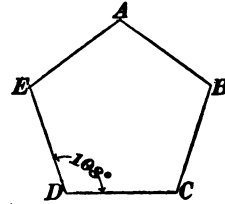


FIG. 34.

Rule 53.—*Multiply 180° by the number of sides less two, and divide the result by the number of sides; the quotient will be the number of degrees in each interior angle.*

EXAMPLE.—If Fig. 34 is a regular pentagon, how many degrees are there in each interior angle?

SOLUTION.—In a pentagon there are five sides; hence, $5 - 2 = 3$ and $180 \times 3 = 540$; $540 \div 5 = 108^\circ$ in each angle. Ans.

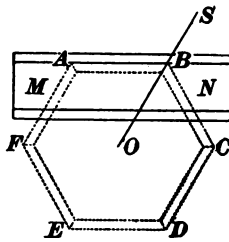


FIG. 35.

EXAMPLE.—It is desired to make a miter-box in which to cut a strip of molding to fit around a column having the shape of a regular hexagon. At what angle should the saw run across the miter-box?

SOLUTION.—In Fig. 35, let AB, BC, CD , etc., represent the pieces of molding as they will fit around the column. First find the size of one of the equal angles of the polygon by the above rule. Number of sides = 6; $6 - 2 = 4$; hence, $180 \times 4 = 720$, and $720 \div 6 = 120^\circ$ in each angle. Now, let MN represent the miter-box, and OS the direction in which the saw should run; then, ABO is the angle made by the saw with the side of the miter-box; but, as the polygon is a regular one, this angle is one-half the interior angle ABC , which we have found to be 120° . Hence, the saw should run at an angle of $\frac{120}{2} = 60^\circ$ with the side of the miter-box.

390. The area of any regular polygon may be found by drawing lines from the center to each angle, and computing the area of each triangle thus formed. Hence, to find the area of any regular polygon:

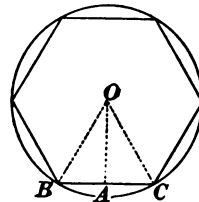


FIG. 36.

Rule 54.—*Multiply the length of a side by half the distance from the side to the center, and that product by the number of sides. The last product will be the area of the figure.*

EXAMPLE.—In Fig. 36 the side BC of the regular hexagon is 12 inches and the distance AO is 10.4 inches; required, the area of the polygon.

SOLUTION.— $10.4 \div 2 = 5.2$; $12 \times 5.2 \times 6 = 374.4$ sq. in. Ans.

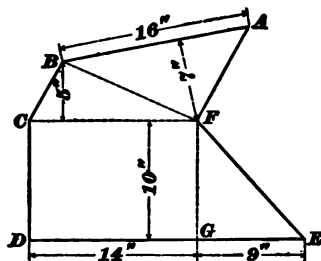


FIG. 37.

391. To obtain the area of any irregular polygon, draw diagonals dividing the polygon into triangles and quadrilaterals, and compute the areas of these separately; their sum will be the area of the figure.

EXAMPLE.—It is required to find the area of the polygon $ABCDEF$, Fig. 37.

SOLUTION.—Draw the diagonals BF and CF and the line FG perpendicular to DE , dividing the figure into the triangles ABF , BCF , and FGE , and the rectangle $FCDG$. Let it be supposed that the altitudes of the figures and the lengths of the sides AB , DG , and GE are as indicated in the polygon above. Then,

$$\text{Area } ABF = \frac{16 \times 7}{2} = 56 \text{ sq. in.}$$

$$\text{Area } BCF = \frac{14 \times 5}{2} = 35 \text{ sq. in.}$$

$$\text{Area } FCDG = 14 \times 10 = 140 \text{ sq. in.}$$

$$\text{Area } FGE = \frac{9 \times 10}{2} = 45 \text{ sq. in.}$$

$$\text{Total area} = 56 + 35 + 140 + 45 = 276 \text{ sq. in. Ans.}$$

EXAMPLES FOR PRACTICE.

1. How many degrees are there in one of the angles of a regular octagon? Ans. 135° .

2. Find the area of the polygon $ABCDEF$ (see Fig. 37), supposing each of the given dimensions to be increased to $1\frac{1}{2}$ times the length given in the figure. Ans. 621 sq. in.

3. What is the area of a regular heptagon, whose sides are 4 inches long, the distance from one side to the center being 4.15 inches? Ans. 58.1 sq. in.

4. At what angle should the saw run in a miter-box to cut strips to fit around the edge of a table top, made in the shape of a regular pentagon? Ans. 54° .

THE CIRCLE.

392. A **circle** (Fig. 38) is a figure bounded by a curved line, called the **circumference**, every point of which is equally distant from a point within, called the **center**.

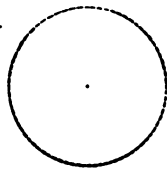


FIG. 38.

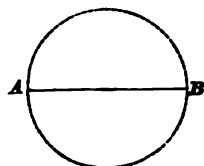


FIG. 39.

393. The **diameter** of a circle is a straight line passing through the center and terminated at both ends by the circumference; thus, AB (Fig. 39) is a diameter of the circle.

394. The **radius** of a circle, AO (Fig. 40), is a straight line drawn from the center O to the circumference. It is equal in length to one-half the diameter. The plural of radius is **radii**, and all radii of a circle are equal.

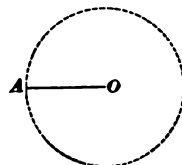


FIG. 40.

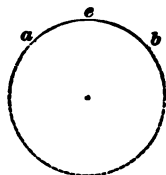


FIG. 41.

395. An **arc** of a circle (see aeb , Fig. 41) is any part of its circumference.

396. A **chord** is a straight line joining any two points in a circumference; or it is a straight line joining the extremities of an arc; thus, the straight line ab , Fig. 42, is a chord of the circle whose corresponding arc is aeb .

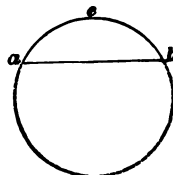


FIG. 42.

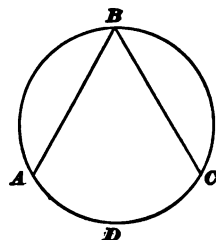


FIG. 43.

397. An **inscribed angle** is one whose vertex lies on the circumference of a circle, and whose sides are chords. It is measured by one-half the intercepted arc. Thus, in Fig. 43, ABC is an inscribed angle, and it is measured by one-half the arc ADC .

EXAMPLE.—If in Fig. 43, the arc $ADC = \frac{1}{3}$ of the circumference, what is the measurement of the inscribed angle ABC ?

SOLUTION.—Since the angle is an inscribed angle, it is measured by one-half the intercepted arc, or $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the circumference. The whole circumference = 360° ; hence, $360^\circ \times \frac{1}{6} = 72^\circ$; therefore, angle ABC is an angle of 72° .

398. If a circle is divided into halves, each half is called a **semicircle**, and each half circumference is called a **semi-circumference**.

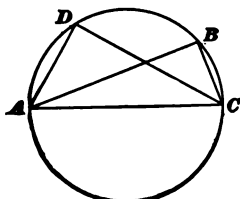


FIG. 44.

Any angle inscribed in a semicircle is a right angle, since it is measured by one-half a semi-circumference, or $180^\circ \div 2 = 90^\circ$. Thus, the angles ADC and ABC , Fig. 44, are right angles, since they are inscribed in a semicircle.

399. An **inscribed polygon** is one whose vertexes lie on the circumference of a circle, and whose sides are chords, as $ABCDE$, Fig. 45.

The sides of an inscribed regular hexagon have the same length as the radius of the circle.

If, in any circle, a radius be drawn perpendicular to any chord, it bisects (cuts in halves) the chord. Thus, if the radius OC , Fig. 46, is perpendicular to the chord AB , $AD = DB$.

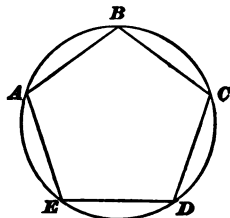


FIG. 45.

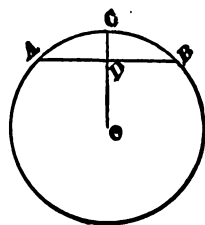


FIG. 46.

EXAMPLE.—If a regular pentagon be inscribed in a circle, and a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side, the perimeter of the pentagon being 27 inches?

SOLUTION.—A pentagon has five sides, and since it is a regular pentagon, all the sides are of equal lengths; the perimeter of a pentagon, which equals the distance around it, or equals the sum of all the sides, is 27 inches. Therefore, the length of one side = $27 \div 5 = 5\frac{2}{5}$ inches. Since the pentagon is an inscribed pentagon, its sides are chords, and as a radius perpendicular to a chord bisects it, we have $5\frac{2}{5} \div 2 = 2\frac{1}{5}$ inches, which equals the length of each of the parts of the side, cut by a radius perpendicular to it.

400. If, from any point on the circumference of a circle, a perpendicular be let fall upon a given diameter, it will divide the diameter into two parts, one of which will be in the same ratio to the perpendicular as the perpendicular is to the other. That is, the perpendicular will be a *mean proportional* between the two parts.

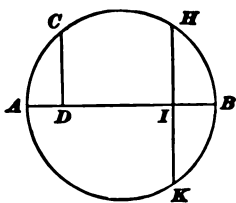


FIG. 41.

If AB , Fig. 47, is the given diameter, and C any point on the circumference, then $AD : CD :: CD : DB$, CD being a mean proportional between AD and DB .

EXAMPLE.—If $HK = 30$ feet, and $IB = 8$ feet, what is the diameter of the circle, HK being perpendicular to AB ?

SOLUTION.— $30 \text{ feet} + 2 \text{ feet} = 15 \text{ feet} = IH$. And $BI : IH :: IH : IA$, or $8 : 15 :: 15 : IA$.

Therefore, $IA = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$ feet, and $IA + IB = 28\frac{1}{8} + 8 = 36\frac{1}{8}$ feet $= AB$ the diameter of the circle. Ans.

401. When the diameter of a circle and the lengths of the two parts into which it is divided are given, the length of the perpendicular may be found by multiplying the lengths of the two parts together and extracting the square root of the product.

EXAMPLE.—In Fig. 47, the diameter of the circle AB is $36\frac{1}{8}$ feet, and the distance BI is 8 feet; what is the length of the line HK ?

SOLUTION.—As the diameter of the circle is $36\frac{1}{8}$ feet, and as BI is 8 feet, IA is equal to $36\frac{1}{8} - 8 = 28\frac{1}{8}$ feet. The two parts, therefore, are 8 and $28\frac{1}{8}$ feet, and their product $= 8 \times 28\frac{1}{8} = 8 \times \frac{225}{8} = 225$; the square root of their product $= \sqrt{225} = 15$ feet, and as $HK = IH + IK$, or $2 IH$, $HK = 15 \times 2 = 30$ feet. Ans.

402. To find the circumference of a circle, the diameter being given:

Rule 55.—*Multiply the diameter by 3.1416.*

EXAMPLE.—What is the circumference of a circle whose diameter is 15 inches?

SOLUTION.— $15 \times 3.1416 = 47.124$ inches. Ans.

403. To find the diameter of a circle, the circumference being given:

Rule 56.—*Divide the circumference by 3.1416.*

EXAMPLE.—What is the diameter of a circle whose circumference is 65.973 inches?

SOLUTION.— $65.973 \div 3.1416 = 21$ inches. Ans.

404. To find the length of an arc of a circle:

Rule 57.—*Multiply the length of the circumference of the circle of which the arc is a part by the number of degrees in the arc, and divide by 360.*

EXAMPLE.—What is the length of an arc of 24° , the radius of the arc being 18 in.?

SOLUTION.— $18 \times 2 = 36$ in. = the diameter of the circle. $36 \times 3.1416 = 113.1$ in., the circumference of the circle of which the arc is a part.

$113.1 \times \frac{24}{360} = 7.54$ in., or the length of the arc. Ans.

405. To find the area of a circle:

Rule 58.—*Square the diameter, and multiply by .7854.*

EXAMPLE.—What is the area of a circle whose diameter is 15 inches?

SOLUTION.— $15^2 = 225$; and $225 \times .7854 = 176.72$ sq. in. Ans.

406. Given the area of a circle, to find its diameter:

Rule 59.—*Divide the area by .7854, and extract the square root of the quotient.*

EXAMPLE.—The area of a circle = 17,671.5 square inches. What is its diameter in feet?

SOLUTION.— $\sqrt{\frac{17,671.5}{.7854}} = 150$ inches.

$\frac{150}{12} = 12\frac{1}{2}$ feet, or the diameter. Ans.

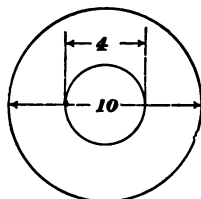


FIG. 48.

EXAMPLE.—What is the area of a flat circular ring, Fig. 48, whose outside diameter is 10 inches, and whose inside diameter is 4 inches?

SOLUTION.—The area of the large circle = $10^2 \times .7854 = 78.54$ square inches; the area of the small circle = $4^2 \times .7854 = 12.57$ square inches. The area of the ring is the difference between these areas, or $78.54 - 12.57 = 65.97$ square inches. Ans.

407. To find the area of a sector (a **sector** of a circle is the area included between two radii and the circumference, as, for example, the area $C O E$, Fig. 13):

Rule 60.—*Divide the number of degrees in the arc of the sector by 360. Multiply the result by the area of the circle of which the sector is a part.*

EXAMPLE.—The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 75° . The diameter of the circle is 12 inches; what is the area of the sector?

SOLUTION.— $\frac{75}{360} = \frac{5}{24}$; and $12^2 \times .7854 = 113.1$ square inches.

$113.1 \times \frac{5}{24} = 23.56$ square inches, the area. Ans.

408. To find the area of a segment of a circle (a **segment** of a circle is the area included between a chord and its arc; for example, the area ABC , Fig. 49):

Rule 61.—*Divide the diameter by the height of the segment; subtract .608 from the quotient, and extract the square root of the remainder. This result multiplied by 4 times the square of the height of the segment, and then divided by 3, will give the area, very nearly.*

The rule, expressed as a formula, is as follows, where D = the diameter of the circle, and h = the height of the segment, see Fig. 49:

$$\text{Area of } ABCA = \frac{4h^3}{3} \sqrt{\frac{D}{h} - .608}.$$

EXAMPLE.—What is the area of the segment of a circle whose diameter is 54 inches, the height of the segment being 20 inches?

SOLUTION.—Substituting in the formula,

$$\begin{aligned} \text{Area} &= \frac{4 \times 20^3}{3} \sqrt{\frac{54}{20} - .608}. \quad \frac{4 \times 20^3}{3} = \frac{4 \times 400}{3} = \\ &= \frac{1,600}{3}; \quad \sqrt{\frac{54}{20} - .608} = \sqrt{2.092} = 1.447; \quad \frac{1,600}{3} \times 1.447 = 771.7 \text{ sq. in.} \end{aligned}$$

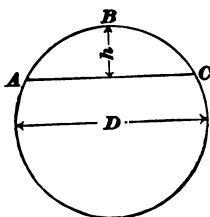


FIG. 49.

Ans.

EXAMPLES FOR PRACTICE.

1. An angle inscribed in a circle intercepts one-third of the circumference. How many degrees are there in the angle? Ans. 60° .

2. Suppose that in Fig. 47, the diameter $AB = 15$ feet, and the distance $BI = 8$ feet. What is the length of the line HK ? Ans. 12 ft.

3. The diameter of a fly-wheel is 18 feet. What is the distance around it to the nearest 16th of an inch? Ans. 56 ft. $6\frac{9}{16}$ in.
4. A carriage wheel was observed to make $71\frac{3}{8}$ turns while going 300 yards. What was its diameter? Ans. 4 ft., nearly.
5. What is the length of an arc of 64° , the radius of the arc being 30 inches? Ans. 33.51 in.
6. Find the area of a circle 2 feet 3 inches in diameter. Ans. 3.976 sq. ft.
7. What must be the diameter of a circle to contain 100 square inches? Ans. 11.28 in.
8. Compute the area of a segment, whose height is 11 inches, and the radius of whose arc is 21 inches. Ans. 289.04 sq. in.
9. Find the area of a flat circular ring whose outside diameter is 12 inches and whose inside diameter is 6 inches. Ans. 84.82 sq. in.

THE PRISM AND CYLINDER.

409. A **solid**, or body, has three dimensions: length, breadth, and thickness. The sides which enclose it are called the **faces**, and their intersections are called **edges**.

410. A **prism** is a solid whose ends are equal and parallel polygons, and whose sides are parallelograms. Prisms take their names from the form of their bases. Thus, a triangular prism is one having a triangle for its base; a hexagonal prism is one having a hexagon for its base, etc.

411. A **cylinder** is a body of uniform diameter whose ends are equal parallel circles.

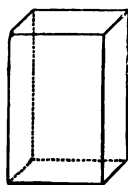


FIG. 50.

412. A **parallelepipedon** (Fig. 50) is a prism whose bases (ends) are parallelograms.

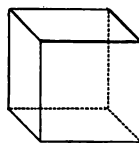


FIG. 51.

413. A **cube** (Fig. 51) is a prism whose faces and ends are squares. All the faces of a cube are equal.

In the case of plane figures, we have had to do with perimeters and areas. In the case of solids, we have to do with the areas of their outside surfaces, and with their contents or volumes.

414. The **entire surface** of any solid is the area of the whole outside of the solid, including the ends.

The **convex surface** of a solid is the same as the entire surface, except that the areas of the ends are not included.

415. A **unit of volume** is a cube each of whose edges is equal in length to the unit. The **volume** is expressed by the number of times it will contain a *unit of volume*.

Thus, if the unit of length is 1 inch, the unit of volume will be the cube whose edges each measure 1 inch, this cube being 1 *cubic inch*; and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be 1 *cubic foot*, etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

Instead of the word *volume*, the expression **cubical contents** is sometimes used.

416. To find the area of the convex surface of a prism or cylinder:

Rule 62.—*Multiply the perimeter of the base by the altitude.*

EXAMPLE.—A block of marble is 24 inches long, and its ends are 9 inches square. What is the area of its convex surface?

SOLUTION.— $9 \times 4 = 36$ = the perimeter of the base; $36 \times 24 = 864$ sq. in., the convex area. Ans.

To find the entire area of the outside surface, add the areas of the two ends to the convex area. Thus, the area of the two ends = $9 \times 9 \times 2 = 162$ sq. in.; $864 + 162 = 1,026$ sq. in. Ans.

EXAMPLE.—How many square feet of sheet iron will be required for a pipe $1\frac{1}{2}$ feet in diameter and 10 feet long, neglecting the amount necessary for lapping?

SOLUTION.—The problem is to find the convex surface of a cylinder $1\frac{1}{2}$ feet in diameter and 10 feet long. The perimeter, or circumference, of the base = $1\frac{1}{2} \times 3.1416 = 4.712$ feet. The convex surface = $4.712 \times 10 = 47.12$ sq. ft. of metal. Ans.

417. To find the volume of a prism or a cylinder:

Rule 63.—*Multiply the area of the base by the altitude.*

EXAMPLE.—What is the weight of a length of wrought-iron shafting 16 feet long and 2 inches in diameter? Wrought iron weighs .28 pound per cubic inch.

SOLUTION.—The shaft is a cylinder 16 feet long. The area of one end, or the base, $= 2^2 \times .7854 = 3.1416$ sq. in. Since the weight of the iron is given per cubic inch, the contents of the shaft must be found in cubic inches. The length, 16 feet, reduced to inches $= 16 \times 12 = 192$ in.; $3.1416 \times 192 = 603.19$ cu. in. = the volume. The weight $= 603.19 \times .28 = 168.89$ lb. Ans.

EXAMPLE.—Find the cubical contents of a hexagonal prism, Fig. 52, 12 inches long, each edge of the base being one inch long.

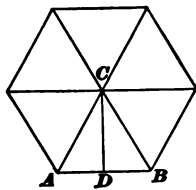


FIG. 52.

SOLUTION.—In order to obtain the area of one end, the distance CD from the center C to one side must be found.

In the right-angled triangle CDA , side $AD = \frac{1}{2} AB$, or one-half inch, and since the polygon is a hexagon, side $CA =$ distance AB , or one inch. Hence, CA being the hypotenuse, the length of side $CD = \sqrt{1^2 - (\frac{1}{2})^2} = \sqrt{1^2 - .5^2} = \sqrt{.75}$, or

.866 inch. Area of triangle $ACB = \frac{1 \times .866}{2} = .433$ sq. in.; area of the whole polygon $= .433 \times 6 = 2.598$ sq. in. Hence, the contents of the prism $= 2.598 \times 12 = 31.176$ cu. in. Ans.

EXAMPLE.—It is required to find the number of cubic feet of steam space in the boiler shown in Fig. 53. The boiler is 16 feet long between

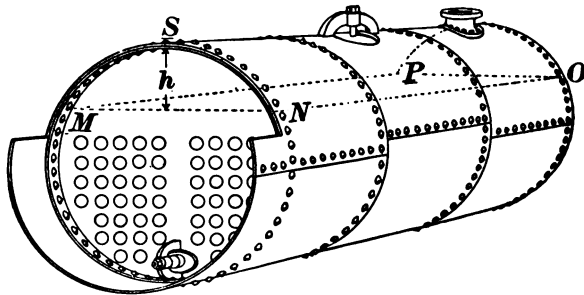


FIG. 53.

heads, 54 inches in diameter, and the mean water line MN is at a distance of 16 inches from the top of the boiler. The volume of the steam outlet casting may be neglected.

SOLUTION.—The volume of the steam space, which is that space within the boiler above the surface $MNOP$ of the water, is found by the rule for finding the volume of a prism or cylinder, the area MNS being the base, and the length NO the altitude. First obtain the area of the segment MNS , whose height h is 16 inches, in square feet; then multiply the result by 16, the length of the boiler.

By the formula previously given, the area of the segment =

$$\frac{4 \frac{4}{3}}{3} \sqrt{\frac{D}{k} - .608} = \frac{4 \times 16^2}{3} \sqrt{\frac{54}{16} - .608};$$

$$\frac{4 \times 16^2}{3} = 341.33; \sqrt{\frac{54}{16} - .608} = \sqrt{2.767} = 1.663.$$

Hence, the area = $341.33 \times 1.663 = 567.63$ sq. in. This, reduced to square feet, = $567.63 \div 144 = 3.942$ sq. ft., and the volume, therefore, = $3.942 \times 16 = 63.07$ cu. ft. Ans.

In the above solution, the space occupied by the stays is not considered, for sake of simplicity. They are not shown in the figure.

EXAMPLE.—In the above boiler there are 60 tubes, $3\frac{1}{2}$ inches outside diameter. How many gallons of water will it take to fill the boiler up to the mean water level, there being 231 cubic inches in a gallon?

SOLUTION.—Find the volume in cubic inches of that part of the boiler below the surface of the water *MNO P*, since the contents of a gallon is given in cubic inches, and from it subtract the volume of the tubes in cubic inches.

This may be done by first finding the *total* area of one end of the boiler in square inches, from it subtracting the area of the segment *MNS*, and the areas of the ends of the tubes in square inches, and then by multiplying the result by the length of the boiler *in inches*.

Total area of one end = $54^2 \times .7854 = 2,290.23$ sq. in.

Area of segment *MNS*, as found in last example, = 567.63 sq. in.

Area of the end of one tube = $3.25^2 \times .7854 = 8.2958$ sq. in.

Area of the ends of the 60 tubes = $8.2958 \times 60 = 497.75$ sq. in.

Hence, the area to be subtracted = $567.63 + 497.75 = 1,065.38$ sq. in. Subtracting, $2,290.23 - 1,065.38 = 1,224.85$ sq. in. = net area.

The cubical contents = $1,224.85 \times 16 \times 12 = 235,171.2$ cu. in. This, divided by 231, will give the number of gallons; whence, $235,171.2 \div 231 = 1,018.06$ gallons of water. Ans.

EXAMPLES FOR PRACTICE.

1. Find the area in square inches of the convex surface of a bar of iron $4\frac{1}{2}$ inches in diameter, and 8 feet 5 inches long. Ans. 1,348.53 sq. in.

2. Find the area of the entire surface of the above bar.

Ans. 1,376.9 sq. in.

3. What is the area of the entire surface of the hexagonal prism whose base is shown in Fig. 52?

Ans. 77.196 sq. in.

4. A multitubular boiler has the following dimensions: diameter, 50 inches; length between heads, 15 feet; number of tubes, 56; outside diameter of tubes, 3 inches; distance of mean water line from top of boiler, 16 inches. (a) Compute the steam space in cubic feet. (b) Find the number of gallons of water required to fill the boiler up to the mean water line.

Ans. $\left\{ \begin{array}{l} (a) 56.4 \text{ cu. ft.} \\ (b) 800 \text{ gallons.} \end{array} \right.$

THE PYRAMID AND CONE.

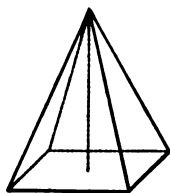


FIG. 54.

418. A **pyramid** (Fig. 54) is a solid whose base is a polygon, and whose sides are triangles uniting at a common point, called the **vertex**.

419. A **cone** (Fig. 55) is a solid whose base is a circle and whose convex surface tapers uniformly to a point called the **vertex**.

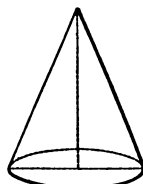


FIG. 55.

420. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

421. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height of a *cone* is any straight line drawn from the vertex to the circumference of the base.

422. To find the convex area of a pyramid or cone:

Rule 64.—*Multiply the perimeter of the base by one-half the slant height.*

EXAMPLE.—What is the convex area of a pentagonal pyramid, if one side of the base measures 6 inches, and the slant height = 14 inches?

SOLUTION.—The base of a pentagonal pyramid is a pentagon, and, consequently, has five sides.

$6 \times 5 = 30$ inches, or the perimeter of the base. $30 \times \frac{14}{2} = 210$ sq. in.,
or the convex area. Ans.

EXAMPLE.—What is the entire area of a right cone whose slant height is 17 inches, and whose base is 8 inches in diameter?

SOLUTION.—The perimeter of the base = $8 \times 3.1416 = 25.1328$ in.

$$\text{Convex area} = 25.1328 \times \frac{17}{2} = 213.63 \text{ sq. in.}$$

$$\text{Area of base} = 8^2 \times .7854 = 50.27 \text{ sq. in.}$$

$$\text{Entire area} = \overline{263.90} \text{ sq. in.} \quad \text{Ans.}$$

423. To find the volume of a pyramid or cone:

Rule 65.—*Multiply the area of the base by one-third of the altitude.*

EXAMPLE.—What is the volume of a triangular pyramid, each edge of whose base measures 6 inches, and whose altitude is 8 inches?

SOLUTION.—Draw the base as shown in Fig. 56; it will be an equilateral triangle, all of whose sides are 6 inches long.

Draw a perpendicular BD from the vertex to the base; it will divide the base into two equal parts, since an equilateral triangle is also isosceles, and will be the altitude of the triangle. In order to obtain the area of the base, this altitude must be determined.

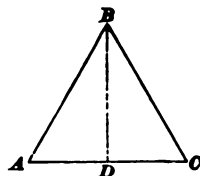


FIG. 56.

In the right-angled triangle BDA , the hypotenuse $BA = 6$ inches, and side $AD = 3$ inches, to find the other side,

$$BD = \sqrt{6^2 - 3^2} = 5.2 \text{ inches, nearly.}$$

$$\begin{aligned} \text{Area of the base, or } BAC, &= \frac{6 \times 5.2}{2} = 15.6 \text{ sq. in.} \quad \text{Hence, the volume} \\ &= 15.6 \times \frac{8}{3} = 41.6 \text{ cu. in.} \quad \text{Ans.} \end{aligned}$$

EXAMPLE.—What is the volume of a cone whose altitude is 18 inches, and whose base is 14 inches in diameter?

SOLUTION.—Area of the base $= 14^2 \times .7854 = 153.94$ sq. in. Hence, the volume $= 153.94 \times \frac{18}{3} = 923.64$ cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. Find the convex surface of a square pyramid whose slant height is 28 inches, and one edge of whose base is $7\frac{1}{2}$ inches long.

Ans. 420 sq. in.

2. What is the volume of a triangular pyramid, one edge of whose base measures 3 inches, and whose altitude is 4 inches?

Ans. 5.2 cu. in.

3. Find the volume of a cone whose altitude is 12 inches, and the circumference of whose base is 31.416 inches.

Ans. 314.16 cu. in.

NOTE.—Find the diameter of the base and then its area.

THE FRUSTUM OF A PYRAMID OR CONE.

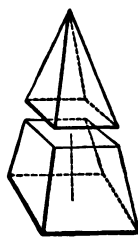


FIG. 57.

424. If a pyramid be cut by a plane, parallel to the base, so as to form two parts, as in Fig. 57, the lower part is called the **frustum** of the pyramid.

If a cone be cut in a similar manner, as in Fig. 58, the lower part is called the **frustum** of the cone.

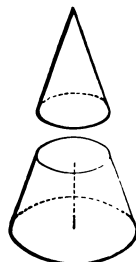


FIG. 58.

425. The upper end of the frustum of a pyramid or cone is called the **upper base**, and the lower end the **lower base**. The **altitude** of a frustum is the perpendicular distance between the bases.

426. To find the convex surface of a frustum of a pyramid or cone:

Rule 66.—*Multiply one-half the sum of the perimeters of the two bases by the slant height of the frustum.*

EXAMPLE.—Given, the frustum of a triangular pyramid, in which one side of the lower base measures 10 inches, one side of the upper base measures 6 inches, and whose slant height is 9 inches; find the area of the convex surface.

SOLUTION.—10 in. $\times 3 = 30$ in., the perimeter of the lower base.

6 in. $\times 3 = 18$ in., the perimeter of the upper base.

$$\frac{30 + 18}{2} = 24 \text{ in., or one-half the sum of the perimeters of the two}$$

bases. $24 \times 9 = 216$ sq. in., the convex area. **Ans.**

EXAMPLE.—If the diameters of the two bases of a frustum of a cone are 12 inches and 8 inches, respectively, and the slant height is 12 inches, what is the entire area of the frustum?

SOLUTION.— $\frac{(12 \times 3.1416) + (8 \times 3.1416)}{2} \times 12 = 376.99$ sq. in., the area of the convex surface.

Area of the upper base = $8^2 \times .7854 = 50.27$ sq. in.

Area of the lower base = $12^2 \times .7854 = 113.1$ sq. in.

The entire area of the frustum = $376.99 + 50.27 + 113.1 = 540.36$ sq. in. **Ans.**

427. To find the volume of the frustum of a pyramid or cone:

Rule 67.—*Add together the areas of the upper and lower bases, and the square root of the product of the two areas; multiply the sum by one-third of the altitude.*

EXAMPLE.—Given, a frustum of a square pyramid (one whose base is a square); each edge of the lower base is 12 inches, each edge of the upper base is 5 inches, and its altitude is 16 inches; what is its volume?

SOLUTION.—Area of upper base = $5 \times 5 = 25$ sq. in.; area of lower base = $12 \times 12 = 144$ sq. in.; the square root of the product of the two areas = $\sqrt{25 \times 144} = 60$. Adding these three results, and multiplying by one-third the altitude, $25 + 144 + 60 = 229$; $229 \times \frac{16}{3} = 1,231\frac{1}{3}$ cu. in. = the volume. **Ans.**

EXAMPLE.—How many gallons of water will a round tank hold, which is 4 feet in diameter at the top, 5 feet in diameter at the bottom, and 8 feet deep?

SOLUTION.—There are 231 cubic inches in a gallon, and the volume of the tank should be found in cubic inches. The tank is in the shape of the frustum of a cone. The upper diameter = $4 \times 12 = 48$ inches; the lower diameter = $5 \times 12 = 60$ inches, and the depth = $8 \times 12 = 96$ inches. Area of upper base = $48^2 \times .7854 = 1,809.56$ sq. in.; area of lower base = $60^2 \times .7854 = 2,827.44$ sq. in.; $\sqrt{1,809.56 \times 2,827.44} = 2,261.95$.

Whence, $1,809.56 + 2,827.44 + 2,261.95 = 6,898.95$; $6,898.95 \times \frac{96}{3} = 220,766.4$ cu. in. = contents. Now, since there are 231 cu. in. in one gallon, the tank will hold $220,766.4 \div 231 = 955.7$ gallons, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. Find the convex surface of the frustum of a square pyramid, one edge of whose lower base is 15 inches long, one edge of whose upper base is 14 inches long, and whose slant height is one inch.

Ans. 58 sq. in.

2. Find the volume of the above frustum, supposing its altitude to be 3 inches.

Ans. 631 cu. in.

3. Find the volume of the frustum of a cone whose altitude is 12 feet and the diameters of whose upper and lower bases are 8 and 10 feet, respectively.

Ans. 766.55 cu. ft.

4. If a tank had the dimensions of example 3, how many gallons would it hold?

Ans. 5,734.2 gallons, nearly.

THE SPHERE AND CYLINDRICAL RING.

428. A **sphere** (Fig. 59) is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called the center.

The word **ball**, or **globe**, is generally used instead of sphere.



FIG. 59.

429. To find the area of the surface of a sphere:

Rule 68.—Square the diameter and multiply the result by 3.1416.

EXAMPLE.—What is the area of the surface of a sphere whose diameter is 14 inches?

SOLUTION.—Diameter squared $\times 3.1416 = 14^2 \times 3.1416 = 14 \times 14 \times 3.1416 = 615.75$ sq. in. Ans.

From this it will be seen that the surface of a sphere equals the circumference of a great circle multiplied by the diameter, a rule often used; a *great circle* of a sphere is the intersection of its surface with a plane passing through its center; for instance, the *great circle* of a sphere 6 in. diameter is a circle of 6 in. diameter. Any number of *great circles* could be described on a given sphere.

430. To find the volume of a sphere:

Rule 69.—*Cube the diameter and multiply the result by .5236.*

EXAMPLE.—What is the weight of a lead ball 12 inches in diameter, a cubic inch of lead weighing .41 pound?

SOLUTION.—Diameter cubed $\times .5236 = 12 \times 12 \times 12 \times .5236 = 904.78$ cu. in., or the volume of the ball. The weight, therefore, $= 904.78 \times .41 = 370.96$ pounds. Ans.

431. To find the convex area of a cylindrical ring:

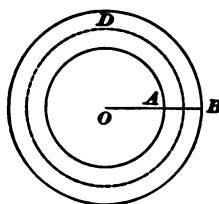


FIG. 60.

A **cylindrical ring** (Fig. 60) is a cylinder bent to a circle. The **altitude** of the cylinder before bending is the same as the length of the dotted center line *D*. The **base** will correspond to a cross-section on the line *AB* drawn from the center *O*. Hence, to find the convex area:

Rule 70.—*Multiply the circumference of an imaginary cross-section on the line *AB*, by the length of the center line *D*.*

EXAMPLE.—If the outside diameter of the ring is 12 inches, and the inside diameter is 8 inches, what is its convex area?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$, and $10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the inner circle is 4 inches; of the outside circle, 6 inches; therefore, the diameter of the cross-section on the line *AB* is 2 inches. Then, $2 \times 3.1416 = 6.2832$ inches, and $6.2832 \times 31.416 = 197.4$ sq. in., the convex area. Ans.

432. To find the volume of a cylindrical ring:

Rule 71.—*The volume will be the same as that of a cylinder whose altitude equals the length of the dotted center line D , and whose base is the same as a cross-section of the ring on the line AB , drawn from the center O . Hence, to find the volume of a cylindrical ring, multiply the area of an imaginary cross-section on the line AB , by the length of the center line D .*

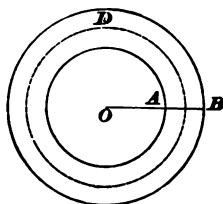


FIG. 61.

EXAMPLE.—What is the volume of a cylindrical ring whose outside diameter is 12 inches, and whose inside diameter is 8 inches?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters = $\frac{12 + 8}{2} = 10$.

$10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the outside circle = 6 inches; of the inside circle = 4 inches; therefore, the diameter of the cross-section on the line AB = 2 inches.

Then, $2^2 \times .7854 = 3.1416$ sq. in., the area of the imaginary cross-section.

And $3.1416 \times 31.416 = 98.7$ cubic inches, the volume. Ans.

EXAMPLES FOR PRACTICE.

1. What is the volume of a sphere 30 inches in diameter?
Ans. 14,137.2 cu. in.
2. How many square inches in the surface of the above sphere?
Ans. 2,827.44 sq. in.
3. Required the area of the convex surface of a circular ring, the outside diameter of the ring being 10 inches, and the inside diameter $7\frac{1}{2}$ inches.
Ans. 107.95 sq. in.
4. Find the cubical contents of the ring in the last example.
Ans. 33.73 cu. in.
5. The surface of a sphere contains 814.16 square inches. What is the volume of the sphere?
Ans. 528.6 cu. in.

1

ELEMENTARY ALGEBRA AND TRIGONOMETRIC FUNCTIONS.

ELEMENTS OF ALGEBRA.

USE OF LETTERS.

433. In arithmetic, numbers are represented by the figures 1, 2, 3, 4, etc. There is no reason, however, why numbers may not be represented by other symbols, such as letters, if rules are provided for their use.

434. In algebra, numbers are represented by both figures and letters. It will be seen later that the use of letters often simplifies the solution of examples and shortens calculations.

435. The principal advantage of letters is that they are general in their meaning. Thus, unlike figures, the letter *a* does not stand for the number 1, the letter *b* for 2, *c* for 3, etc., but *any* letter may be taken to represent *any* number, it being only necessary that a letter shall always stand for the same number *in the same example*.

436. To illustrate this difference between letters and figures, we may take an example, as follows: If a farmer exchanges 20 bushels of oats, worth 40 cents per bushel, for 8 bushels of wheat, what is the price of the wheat per bushel? A rule for solving this problem, and others like it, would be as follows: *Multiply the number of bushels of oats by the price per bushel, and divide the result by the number of bushels of wheat.*

This rule is *general*, because it tells us what to do with the number of bushels and with the prices of the oats and the wheat, *whatever they may be*.

A more concise way of stating this rule is to use letters in the same manner as in formulas. Thus :

Let a = number of bushels of oats ;
 b = price per bushel of oats ;
 c = number of bushels of wheat ;
 d = price per bushel of wheat.

Then, according to the rule,

$$\frac{\text{bushels of oats} \times \text{price per bushel}}{\text{bushels of wheat}} = \text{price of wheat},$$

or
$$\frac{a \times b}{c} = d.$$

In the example in question, $a = 20$, $b = 40$, and $c = 8$. Hence, writing for a , b , and c their values, 20, 40, and 8, d , the price per bushel of wheat $= \frac{20 \times 40}{8} = 100$. Here the expression $\frac{20 \times 40}{8}$ corresponds to $\frac{a \times b}{c}$, but this difference is to be noticed : $\frac{20 \times 40}{8}$ applies only to *this* example, and by performing the operations indicated only one answer can be obtained, while $\frac{a \times b}{c}$ is *general* in its application, in the same way that the rule previously given is general. That is, while a , b , and c stand for the numbers 20, 40, and 8 in *this* example, they may stand for other numbers in *another* example ; hence, by writing their values in place of the letters, and by performing the operations indicated, the answer to any example of the same kind may be obtained. Consequently, while figures or combinations of figures always represent the same numbers, letters are more general, and may represent any numbers, according to the conditions of the example.

437. An **equation** is a statement of equality between two expressions. Thus, $x + y = 8$ is an equation, and

means that the sum of the numbers represented by x and y is equal to 8. Examples are solved in algebra by the aid of equations, in which numbers are represented both by letters and by figures. The following simple example will give an idea of the method of solution :

EXAMPLE.—If an iron rail 30 feet long is cut in two so that one part is four times as long as the other, how long is the shorter part ?

SOLUTION.—Since any letter may represent any number,

Let x = the length of the shorter part.

Then, $4 \times x$ (written $4x$) = the length of the longer part.

But the sum of the two parts must equal the total length, 30 feet.

Hence, $x + 4x = 30$.

Adding x and $4x$, $5x = 30$.

Whence, dividing by 5, $x = 6$ feet. Ans.

438. The student has probably noticed the similarity between an equation and a **formula**. All formulas are equations, and the same rules apply to both. An equation is not called a formula, however, unless it is a statement of a general rule.

439. Algebra treats of the equation and its use. Since the use of equations involves the use of letters, it will be necessary before considering equations to take up addition, subtraction, multiplication, etc., of expressions in which letters are used.

NOTATION.

440. The term **quantity** is used to designate any number that is to be subjected to mathematical processes. A quantity is strictly a concrete number ; as, 6 books, 5 pounds, 10 yards. *Symbols* used to *represent* numbers, and expressions containing two or more such symbols, as a , x , bd , 10, $(c + 12)$, etc., are often called *quantities*, the term being a convenient one to use.

441. The **signs** $+$, $-$, $\times$, $\div$ are the same in algebra as in arithmetic. The sign of multiplication $\times$ is usually omitted, however, multiplication being indicated by simply writing the quantities together. Thus, abc means $a \times b \times c$;

$2xy$ means $2 \times x \times y$. Evidently the sign can not be omitted between two *figures*, as addition instead of multiplication would then be indicated. Thus, 24 means $20 + 4$ instead of 2×4 .

442. A **coefficient** is a figure or letter prefixed to a quantity ; it shows how many times the latter is to be taken. Thus, in the expression $4a$, 4 is the coefficient of a , and indicates that a is to be taken four times ; that is, $4a$ is equal to $a + a + a + a$. When several quantities are multiplied together, any of them may be regarded as the coefficient of the others. Thus, in $6axy$, 6 is the coefficient of axy ; $6a$, of xy ; $6ax$, of y , etc. In general, however, when a coefficient is spoken of, the numerical coefficient only is meant, as the 6 above. When no numerical coefficient is written it is understood to be 1 . Thus, cd is the same as $1cd$.

443. The **factors** of a quantity are the quantities which, when multiplied together, will produce it. Thus, 2 , 3 , and 3 are the factors of 18 , since $2 \times 3 \times 3 = 18$; 2 , a , and b are the factors of $2ab$, since $2 \times a \times b = 2ab$.

444. An **exponent** is a small figure placed at the right and a little above a quantity ; it shows how many times the latter is to be taken *as a factor*. Thus, $4^3 = 4 \times 4 \times 4 = 64$, the exponent 3 showing that the number 4 is to be used three times as a factor ; likewise $a^5 = aaaaa$. Any quantity written without an exponent is understood to have the exponent 1 ; thus, $b^1 = b$.

445. The difference between a coefficient and an exponent should be clearly understood. A coefficient *multiplies* the quantity which it precedes ; it shows that the quantity is to be *added to itself*. Thus, $3a = 3 \times a$, or $a + a + a$. An exponent indicates that a quantity is to be *multiplied by itself*. Thus, $a^3 = a \times a \times a$. A more complete definition of an exponent will be given later.

446. A **power** is the result obtained by taking a quantity two or more times *as a factor*. For example, 16 is

the fourth power of 2, because 2 multiplied by itself until it has been taken four times as a factor produces 16 ; a^3 is the third power of a , because $a \times a \times a = a^3$.

447. A **root** of a quantity is one of its equal factors. Thus, 2 is the root of 4, 8, and 16, since $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, and $2 \times 2 \times 2 \times 2 = 16$, 2 being one of the equal factors in each case. In like manner, a is a root of a^2 , a^3 , a^4 , etc. The symbol which denotes that the second, or square, root is to be extracted is $\sqrt{\quad}$; it is called the **radical sign**, and the quantity under the sign is called the **radical**. For other roots the same symbol is used, but with a figure, called the *index* of the root, written above it to indicate the root. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, etc., signify the square root, cube root, fourth root, etc., of a .

448. The use of the parenthesis, bracket, brace, and vinculum is explained in Art. 341. These symbols are called **symbols of aggregation**, meaning that the quantities enclosed within them are aggregated, or collected, into one quantity.

449. The **terms** of an algebraic expression are those parts which are connected by the signs + and -. Thus, x^2 , $-2xy$, and y^2 are terms of the expression $x^2 - 2xy + y^2$. When a term contains both figures and letters, the part consisting of letters is called the **literal** part of the term ; thus, xy is the literal part of the term $2xy$.

450. **Like terms** are those which differ only in their numerical coefficients ; all others are **unlike terms**. Thus, $2ab^2$ and $5ab^2$ are like terms ; $5ab$ and $5ab^2$ are unlike terms, because one contains b and the other b^2 .

451. A **monomial** is an expression consisting of only one term ; as, $4abc$, $3x^2$, $2ax^3$, etc.

452. A **binomial** is an expression consisting of two terms ; as, $a + b$, $2a + 5b$, etc.

453. A **trinomial** is an expression consisting of three terms ; as, $a^2 + 2ab + b^2$, $(a + x)^2 - 2(a + x)y + y^2$, etc., the expression $(a + x)$ being treated as one quantity. (Art. 341.)

454. A **polynomial** is an expression consisting of more than one term. The name is usually applied only to an expression consisting of four or more terms.

455. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is said to be arranged according to the *increasing powers* of a , because the exponents of a increase in each term from left to right, the exponent of the first a being 1 understood. (Art. **444.**) The polynomial $a^3b^3 + ab^3 + 4a^4b + 1$ is arranged according to the *decreasing powers* of b , the exponents of b decreasing in order from left to right.

456. The arrangement of the terms of a polynomial does not affect its value. Thus, $x^2 + 2xy + y^2$ has the same value as $2xy + y^2 + x^2$, just as $2 + 6 + 4$ has the same value as $6 + 4 + 2$.

READING ALGEBRAIC EXPRESSIONS.

457. Quantities like a , x , b^2 , etc., are read " a ," " x ," " b square," etc. In reading monomials in which multiplication is indicated, the word "times" is not used. Thus, abc is read " abc "; $7ad^2b^3$ is read " $7ad$ square b cube."

458. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is read " a , plus a square b , plus $2a$ cube, minus $3a$ fourth b , minus a fifth." Considerable care is required when reading expressions containing polynomials. Thus, if $4(a - b)$ were read " $4a$ minus b ," the binomial $4a - b$ would be understood. It *should* be read " 4 times $a - b$," or " 4 times the parenthesis a minus b ," in which case it will be understood that 4 multiplies the whole quantity $a - b$, since the word "times" is not used with monomials. Again, $m(m^2 + 2mn + n^2)$ and $m(m^2 + 2mn) + n^2$ should each be so read that there can be no doubt as to whether the n^2 is to be multiplied by m or not.

Let the distinction to be made in reading the following be observed :

$$\sqrt{\frac{m+n}{x-y^2}} \text{ and } \sqrt{m + \frac{n}{x-y^2}}.$$

In the first case, the whole quantity $m + n$ is divided by $x - y^2$, and it would be clear to say, "the square root of the fraction $m + n$ over $x - y^2$." In the second case, where the n only is divided by $x - y^2$, it may be read, "the square root of the quantity, m plus the fraction n over $x - y^2$." The word "quantity" shows that the square root of the whole expression is taken, and the word "fraction" after "plus" shows that only the n is divided by $x - y^2$.

459. When a polynomial is affected by an exponent, it should be indicated clearly. Thus,

$$(3a - d^2) (3a - d)^2 (3a - d^3)^3$$

should be read, " $3a - d$ square, times the square of $3a - d$, times the square of $3a - d$ square."

460. Sometimes expressions like A' , B'' , c' , d'' , C''' , a_2 , etc., appear in formulas or elsewhere in algebraic problems when it is desirable to have the same letter represent different quantities that are similar, or correspond to one another. The marks ', ', ', ', ', etc., serve to distinguish the letters. (See Art. 343.) The expressions are also used to designate similar or corresponding lines in geometrical figures, as will appear in Mechanical Drawing. A' , B'' , C''' , etc., are read "*a major prime, b major second, c major third,*" etc.; a' , b'' , c''' , etc., are read "*a minor prime, b minor second, c minor third,*" etc.; a_1 , B_2 , C_3 , d_4 , etc., are read "*a minor sub-one, b major sub-two, c major sub-three, d minor sub-four,*" etc.

The words *major* and *minor* are used only when capitals and small letters are employed in the same problem. Otherwise they are dropped, and a' , b_2 , for example, are read "*a prime, b sub-two.*"

POSITIVE AND NEGATIVE QUANTITIES.

461. **Positive** and **negative** are terms applied to quantities of opposite character; as, money earned and money owed, water running into a tank and water running out, a distance up-stream and a distance down-stream, the height of a tower and the depth of a well, the pull on a lifting-rope and the weight of the load, etc.

462. Positive quantities are preceded by the sign plus, as $+2xy$, $+ab$, etc., and negative quantities by the sign minus, as $-2xy$, $-ab$, etc. Thus, if money earned is $+\$50$, a like amount owed is $-\$50$. If the quantity of water running into a tank is denoted by $+a$, the same quantity running out should be denoted by $-a$.

463. It really does not matter which quantity be taken as positive and which as negative, so long as the characters of the positive and the negative quantity are opposite; but it is customary to call something gained positive and something lost negative. Thus, money earned is usually regarded as positive, money owed as negative; distance up, positive, distance down, negative.

464. The signs $+$ and $-$ may be used in two entirely different senses; heretofore we have used them exclusively as symbols of operation; thus $+$ placed between two quantities indicates that they are to be added, etc. In the distinction between positive and negative quantities; however, we denote the positive quantity by the sign $+$ and the negative quantity by the sign $-$. Hence, under different circumstances, these signs may denote addition and subtraction, or they may denote positive and negative quantities. Suppose we write the expression $\$500 - \$200 = \$300$; the sign $-$ in this case indicates that the $\$200$ is subtracted from the $\$500$. Suppose, however, that a man has in his possession $\$500$ and owes $\$200$. The former amount we may denote by $+\$500$, and the latter, since it is owed, by $-\$200$; in this case the sign $-$ before the $\$200$ indicates the negative character of the quantity. To find how much the man is worth we add the two, thus:

$$+\$500 + (-\$200) = +\$300.$$

In this addition the second $+$ sign denotes the operation of addition, while the three signs immediately before the quantities denote the positive or negative character of the quantities.

465. It is usual to consider a quantity as increasing in a *positive* direction; *any positive quantity, no matter how small, is always considered greater than any negative quantity, no matter how large.*

The value of a negative quantity is conceived to *increase* as its numerical value *decreases*. A man who is \$10 in debt is better off than one who is \$50 in debt; and the man who has \$5 in the bank is better off than either. Thus, 5 is greater than -10 , and -10 is greater than -50 .

466. When writing algebraic expressions, if a positive term stands alone, or if the first term of an expression is positive, the plus sign is omitted, it being understood that the term is positive. Thus, $3a$ means the same as $+3a$, and $a - b$ the same as $+a - b$. The minus sign must never be omitted. Polynomials are usually written with a positive term first, and monomials with the letters arranged alphabetically.

EXAMPLES FOR PRACTICE.

467. Express the following algebraically:

- Three x square y square, minus two cd into a plus b .
Ans. $3x^2y^2 - 2cd(a + b)$.
- The quantity m square plus two mn plus n square in parenthesis, times a square b cube c fourth.
Ans. $(m^2 + 2mn + n^2)a^2b^3c^4$.
- A , plus the square root of D , times the parenthesis X plus Y .
Ans. $A + \sqrt{D}(X + Y)$.
- A , plus the radical D times the parenthesis X plus Y .
Ans. $A + \sqrt{D}(X + Y)$.
- Ten x plus y , minus seven times the quantity x minus the fraction y over 4 in parenthesis, plus the fraction x square minus y square over two cd .
Ans. $10x + y - 7\left(x - \frac{y}{4}\right) + \frac{x^2 - y^2}{2cd}$.

When $a = 6$, $b = 5$, and $c = 4$, find the numerical values of:

- $a^2 + 2ab + b^2$.
Ans. $6^2 + 2 \times 6 \times 5 + 5^2 = 121$.
- $2a^2 + 3bc - 5$.
Ans. $72 + 60 - 5 = 127$.
- $2ac^2 - a^2(a + b)$.
Ans. 11,892.
- $abc^2 + ab^2c - a^2bc$.
Ans. 360.

When $x = 8$ and $y = 6$, what do the following equal:

10. $(x + y)(x - y) - \sqrt{\frac{x + y^2}{11}}?$

Ans. $(8 + 6)(8 - 6) - \sqrt{\frac{8 + 6^2}{11}} = 26.$

11. $\sqrt[4]{(x + y^2)(x^2 + y)} - (x - y)(\sqrt[4]{x + y})?$

Ans. 39.5.

12. $\frac{x^2y^2}{x + y} + \frac{x^2y(x + y^2)}{\sqrt[4]{8xy}}?$

Ans. 1,572.57.

ADDITION AND SUBTRACTION.

PRELIMINARY IDEAS.

468. Suppose we take a point, as A on the line shown in Fig. 62, and lay off equal distances in opposite directions.

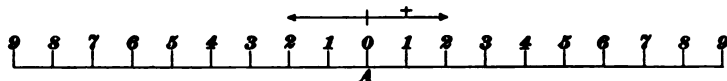


FIG. 62.

Now, let us call distances to the right $+$, or *positive*, and distances to the left $-$, or *negative*. Let us also call a movement to the right positive, and one to the left negative. Suppose the positive direction east, the negative west, and the distances to represent miles.

Suppose a man starts from A and walks east 6 miles, and after a pause walks 3 miles farther east. His distance from A is $+6 + (+3) = +9$ miles, the plus sign being taken because the motion is in the positive direction. Suppose, however, the man starts from A and walks west 3 miles, and after a pause walks 5 miles farther west. His distance from A is 8 miles west of A , or 8 miles in a negative direction; that is $-3 + (-5) = -8$. As a third case, imagine the man to walk 6 miles east, and then turn around and walk 4 miles west. Counting 4 west from 6, we see that he would still be 2 miles east of A , or 2 miles in a positive direction; that is, $+6 + (-4) = 2$. If, instead of walking back 4 miles, he had walked back 10 miles, we find by counting 10 miles west from 6 that he would have been 4 miles west of

4, or 4 miles in a negative direction, $+6 + (-10) = -4$.
For reference, the above results are collected:

$$\begin{aligned} +6 + (+3) &= +9 \\ -3 + (-5) &= -8 \\ +6 + (-4) &= +2 \\ +6 + (-10) &= -4 \end{aligned}$$

The student should observe carefully that in each of these additions the signs immediately before the numbers denote their positive or negative character, while the $+$ sign in front of the parenthesis denotes the operation of addition.

469. From these illustrations we have the following important principle: *If all the terms to be added are positive, the sum is positive; if all are negative, the sum is negative. If one term is positive and the other is negative, the sum has the sign of the numerically greater.* If there are several terms to be added, part of which are positive and part negative, the sum is positive if the sum of the positive terms is numerically greater than the sum of the negative terms.

When the terms have the same sign, the numerical sum is that which would be obtained if the signs were disregarded. Thus, in the first case above, the signs of 6 and 3 are the same, therefore the sum is numerically $6 + 3 = 9$; likewise in the second, the signs of the 3 and the 5 are alike and the sum is numerically $3 + 5 = 8$. When, however, one term is positive and the other is negative, the sum is the numerical difference between the terms. Thus, in the third case above, the signs of 6 and 4 are different, and the sum, 2, is the numerical difference; likewise in the fourth case, the 6 and the 10 have different signs, and the sum, 4, is the numerical difference.

470. To add like terms containing letters, we simply add the coefficients, having regard for the proper signs, and annex the literal part. Thus, the sum of $6ax^2y$ and $3ax^2y$ is $9ax^2y$; the sum of $3ab$ and $-11ab$ is $-8ab$; and the sum of $9m^2n$, $3m^2n$, $-8m^2n$, and $2m^2n$ is $6m^2n$.

ADDITION OF MONOMIALS.

471. Like Quantities.—To add like quantities having the same sign:

Rule I.—*Add the coefficients, give the sum the common sign, and annex the common literal part.*

To add like quantities having different signs:

Rule II.—*Add the positive and the negative coefficients separately, and from the greater sum subtract the less. Give the remainder the sign of the greater sum, and annex the common literal part.*

EXAMPLE.—Find the sum of $-2abxy$, $-abxy$, $-3abxy$, and $-6abxy$.

SOLUTION.—The sum of the coefficients is 12 (remember that the coefficient of $-abxy$ is 1), and the common sign is $-$. The common literal part, $abxy$, annexed to these gives as the result $-12abxy$. (Rule I.)

EXAMPLE.—Combine xy^2 , $-2xy^2$, $8xy^2$, and $-4xy^2$.

SOLUTION.—The sum of the coefficients of the positive terms is 9, and of the negative terms, 6. Their difference is 3, and the sign of the greater sum is $+$. The common literal part, xy^2 , annexed to these gives as the result $3xy^2$. (Rule II.)

SUBTRACTION OF MONOMIALS.

472. Referring again to Fig. 62, suppose two men, C and D , to start from point A and travel eastward. At the end of a certain time, C has walked 8 miles and D has walked 5 miles. Now, the distance between C and D is 3 miles; to pass from $+5$ to $+8$, we must walk $+3$ miles east, or in a *positive* direction; or $+8 - (+5) = +3$. Observe that the minus sign here denotes subtraction, while the three plus signs denote that the three quantities which they precede are positive. The difference between 5 and 8 means how far, and in what direction we must go to pass from 5 to 8 or from 8 to 5. If we pass from $+5$ to $+8$, we move in a positive direction, and we say $+5$ from $+8$ is equal to $+3$. Suppose, however, we pass from $+8$ to $+5$; we move westward, or in a *negative* direction, a distance of 3 miles. Hence, we say that $+8$ from $+5$ is -3 , or $+5$

$-(+8) = -8$. We shall always consider the point we pass *from* as the subtrahend, or quantity to be subtracted, and the point we *approach* as the minuend, or quantity we subtract from.

Suppose C has walked 7 miles east and D has walked 4 miles west, how far apart are they? To pass from D to C we must travel 11 miles east, or in a positive direction. Therefore, $7 - (-4) = +11$. To pass from C to D , we travel 11 miles west, or in a negative direction; $-4 - (+7) = -11$.

The following exercises may be studied in connection with Fig. 62:

From +3 to +8 is +5, or $+8 - (+3) = +5$.

From +10 to +6 is -4, or $+6 - (+10) = -4$.

From -5 to +4 is +9, or $+4 - (-5) = +9$.

From -9 to -2 is +7, or $-2 - (-9) = +7$.

From -3 to -7 is -4, or $-7 - (-3) = -4$.

From +2 to -4 is -6, or $-4 - (+2) = -6$.

473. In every case, the difference is what must be added to the subtrahend to obtain the minuend. Thus, if I am 3 miles east of A , Fig. 62, how far must I go to be 8 miles east of A ? Evidently 5 miles, since 5 miles added to 3 miles gives 8 miles. Similarly, the difference between -5 and +4 is +9, since +9 must be added to -5 to obtain +4; that is, we must travel 9 miles east in passing from -5 to +4.

474. We have seen that if we add +6 and -4 we obtain +2 as the sum. (Art. 468.) If we subtract +4 from +6, the difference is $+6 - (+4) = +2$, since +2 must be added to +4 to make +6. If we add +6 and -10, the sum is -4; if we subtract +10 from +6, the difference is -4. Therefore, it appears that if we wish to subtract one quantity from another, we obtain the same result if we change the sign of the quantity to be subtracted and *add* it to the other quantity. Thus, +7 subtracted from +12 is the same as -7 added to +12, the result being 5 in either case. -3 *mn* subtracted from 4 *mn* gives the

same result as $+3mn$ added to $4mn$. The following exercises are given as illustrations:

$$\begin{aligned} +5 - (-2) &= +5 + (+2) = +7. \\ -6 - (+3) &= -6 + (-3) = -9. \\ +2a - (+a) &= +2a + (-a) = +a. \\ +5xy - (+15xy) &= +5xy + (-15xy) = -10xy. \end{aligned}$$

475. To subtract like quantities:

Rule.—*Change the sign of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $-3ab^2x$ take $7ab^2x$.

SOLUTION.—Changing the sign of the subtrahend, $7ab^2x$, and adding, we have $-3ab^2x + (-7ab^2x) = -10ab^2x$. Ans.

476. Unlike Quantities.—In arithmetic, unlike numbers, as 5 books and 3 dollars, can not be added or subtracted. So, in algebra, unlike terms, as $3ab^2$, $4xy$, $2m$, etc., can not be combined or subtracted, except by indicating the operations by signs.

Expressions in algebra are composed of quantities between which operations of addition, multiplication, etc., are indicated. The trinomial $m^2 - 2mn + n^2$, for example, is the indicated sum of m^2 , $-2mn$, and n^2 , and it is to be considered as one quantity, in the same way that an arithmetical sum, obtained by actually performing the addition, is considered.

EXAMPLE.—What does $7cd^3 - 8cx - cd^3 + 6adx + 2cd^3$ equal?

SOLUTION.—In this case, part of the terms are like and part unlike. Combining like terms, $7cd^3 + 2cd^3 - cd^3 = 8cd^3$. Connecting the unlike terms with this result by their respective signs, we have as the final result $8cd^3 - 8cx + 6adx$. Ans.

EXAMPLE.—Subtract $2m - 3$ from $7m + 2xy$.

SOLUTION.—As in the example above, we have like and unlike terms. Subtracting like terms, Art. 475, we have $7m - 2m = 5m$. We must now connect the unlike terms by their respective signs. Since -3 is in the subtrahend, its sign will be changed, giving us $5m + 2xy + 3$. Ans.

EXAMPLES FOR PRACTICE.

477. Find the sum of the following:

- | | |
|---|----------------|
| 1. $-6a^2$, $2a^2$, $-5a^2$, $4a^2$, $-3a^2$, and a^2 . | Ans. $-7a^2$. |
| 2. $2a^2b$, $-a^2b$, $11a^2b$, $-5a^2b$, $4a^2b$, and $-9a^2b$. | Ans. $2a^2b$. |

3. $2x^2, 3xy, -x^2, 8y^2, -5xy$, and $-7y^2$. Ans. $x^2 - 2xy + y^2$.

NOTE.—Combine like terms and connect with respective signs.

4. $a^2bc, -2ab^2c, 3abc^2, -4a^2bc$, and $5ab^2c$.

Ans. $3ab^2c - 3a^2bc + 3abc^2$.

Solve the following:

5. From $17a$ take $-11a$. Ans. $28a$.

6. From $-11a$ take $17a$. Ans. $-28a$.

7. Subtract $5cd$ from $-4cd$. Ans. $-9cd$.

8. Subtract $-10b^2$ from $-10b^2$. Ans. 0.

9. What quantity added to $10xy$ will produce $-12xy$? Ans. $-22xy$.

10. What, then, does $10xy$ subtracted from $-12xy$ equal? Ans. $-22xy$.

ADDITION AND SUBTRACTION OF POLYNOMIALS.

478. To add polynomials:

Rule.—Write the expressions underneath one another, with like terms in the same vertical column. Add each column separately, and connect the sums by their proper signs.

EXAMPLE.—Find the sum of $5a^2 + 6ac - 3b^2 - 2xy$, $7ac - 3a^2 + 4b^2 + 3xy$, and $4xy - 5b^2 + 8ac - a^2$.

SOLUTION.—Writing like terms in the same vertical column, we have

$$\begin{array}{r} 5a^2 + 6ac - 3b^2 - 2xy \\ - 3a^2 + 7ac + 4b^2 + 3xy \\ - a^2 + 8ac - 5b^2 + 4xy \\ \hline \text{sum} \quad a^2 + 21ac - 4b^2 + 5xy. \quad \text{Ans.} \end{array}$$

EXAMPLE.—Find the sum of $a^2x - ax^2 - x^3$, $ax - x^2 - a^2$, $-2a^2 - 2a^2x - 2ax^2$, and $3a^2 - 3a^2x + 3ax^2$.

SOLUTION.—

$$\begin{array}{r} a^2x - ax^2 - x^3 \\ - x^2 - a^2 + ax \\ - 2a^2x - 2ax^2 - 2a^2 \\ - 3a^2x + 3ax^2 + 3a^2 \\ \hline \text{sum} \quad -4a^2x + 0 - 2x^2 + 0 + ax = \\ \quad ax - 4a^2x - 2x^2. \quad \text{Ans. (Arts. 455 and 456.)} \end{array}$$

479. To subtract one polynomial from another:

Rule.—Write the subtrahend underneath the minuend, with like terms in the same vertical column. Change the sign of each term of the subtrahend, and proceed as in addition.

EXAMPLE.—From $3ac - 2b$ subtract $ac - b - d$.

SOLUTION.—
$$\begin{array}{r} 3ac - 2b \\ - ac + b + d, \text{ subtrahend with signs changed.} \\ \hline \text{difference } 2ac - b + d. \text{ Ans.} \end{array}$$

EXAMPLE.—From $2x^3 - 3x^2y + 2xy^2$ subtract $x^3 - xy^2 + y^3$.

SOLUTION.—
$$\begin{array}{r} 2x^3 - 3x^2y + 2xy^2 \\ - x^3 \qquad \qquad + xy^2 - y^3, \text{ subtrahend with signs changed} \\ \hline \text{difference } x^3 - 3x^2y + 3xy^2 - y^3. \text{ Ans.} \end{array}$$

EXAMPLES FOR PRACTICE.

480. Find the sum of the following:

- $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$.
Ans. $7ax + 3bx + 6by - ay$.
- $a - x + 4y - 3z + w$, $z + 3a - 2x - y - w$, and $x + y + z$.
Ans. $4a - 2x + 4y - z$.
- $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a$, and $2c - 3a + 4b$.
Ans. $3a + 3b + 3c + 3d$.
- $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$.
Ans. $x - y + 3m + 3n$.

Solve the following:

- From $7a + 5b - 3c$ take $a - 7b + 5c - 4$.
Ans. $6a + 12b - 8c + 4$.
- From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.
Ans. $4m - 8n - r + 3s$.
- Subtract $2x - 2y + 2$ from $y - x$.
Ans. $3y - 3x - 2$.
- Subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$ from $5x^3 + x^2y - 6xy^2 + y^3$.
Ans. $2x^3 - 3x^2y + xy^2 + xy^3$.

SYMBOLS OF AGGREGATION.

481. Parentheses, brackets, etc., being used to enclose expressions that are to be treated as one quantity, the sign before the symbol affects the *entire expression*, not the first term only. Thus, $-(a^2 - 2ab + b^2)$ signifies that all the terms are to be subtracted from what precedes, not a^2 only.

482. When combining the terms of any expression without parentheses, we proceed as in addition of monomials. When we have a parenthesis preceded by a minus sign, we must consider the expression within the parenthesis as a

subtrahend, and change all signs before removing the parenthesis.

If, on the contrary, the sign of the parenthesis is plus, we may remove the symbols, but we *must not* change the signs of the expression, because we do not change the signs of an expression to be added.

483. When a quantity is enclosed by a parenthesis, the first term is understood to have the plus sign, unless the minus sign is given; thus, in the expression $-(8x + 5 - 2b)$, the minus sign refers to the whole quantity. The sign of $8x$ is $+$, and the expression if written in full would be $-(+8x + 5 - 2b)$.

EXAMPLE.—Remove the parenthesis from $4c - (3a + 4ab - d)$.

SOLUTION.—Changing the sign of each enclosed term, and remembering that the sign of $3a$ is $+$, understood, we have as the result $4c - 3a - 4ab + d$. Ans.

EXAMPLE.—Remove the parentheses from $4a - 5x - (a - 4x) + (x - 8a)$.

SOLUTION.— $4a - 5x - (a - 4x) + (x - 8a) = 4a - 5x - a + 4x + x - 8a$. Adding the like terms, we have

$$\begin{array}{r} 4a - 5x \\ - a + 4x \\ - 8a + x \\ \hline - 5a + 0 = - 5a. \text{ Ans.} \end{array}$$

484. Symbols of aggregation will often be found enclosing others. In such cases they may be removed in succession, *always beginning with the innermost pair*.

EXAMPLE.—Remove all the symbols of aggregation from $6a - \{b - [7cd - 4a + (2cd - a - b)]\}$.

SOLUTION.—We first remove the vinculum. This being in effect the same as the parenthesis, the minus sign before the a indicates that $+a$ and $-b$ are to be subtracted.

Hence, we have

$$6a - \{b - [7cd - 4a + (2cd - a + b)]\}.$$

Removing the parenthesis we have

$$6a - \{b - [7cd - 4a + 2cd - a + b]\}.$$

This, with the brackets removed, is equal to

$$6a - \{b - 7cd + 4a - 2cd + a - b\},$$

which, in turn, is equal to

$$6a - b + 7cd - 4a + 2cd - a + b.$$

Combining like terms,

$$6a - 4a - a - b + b + 7cd + 2cd = a + 9cd. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

485. Remove the parentheses from the following:

1. $-(2mn - m^2 - n^2)$ Ans. $m^2 - 2mn + n^2$.
2. $1 - (-b + c + 3)$. Ans. $b - c - 2$.
3. $5a - 4b + 3c - (-3a + 2b - c)$. Ans. $8a - 6b + 4c$.
4. $3x - (2x - 5) + (7 - x)$. Ans. 12.

Remove the symbols of aggregation from the following:

5. $m - [4n - k - (m + n - 2k)]$. Ans. $2m - 3n - k$.
6. $5x - (2x - 3y) - (x + 5y)$. Ans. $2x - 2y$.
7. $3a - [7a - (5a - b - a)] - (-a - 4b)$. Ans. $a + 3b$.
8. $3x + \{2y - [5x - (3y + x - 4y)]\}$. Ans. $y - x$.
9. $100x - \{200x - [500x - (-100x) - 300x] - 400x\}$. Ans. $600x$.
10. $7cx - \{4cy - [(4cx + 3cy) + cy - cx]\}$. Ans. $10cx$.

NOTE.—Observe that the sign before the parenthesis is + understood.

MULTIPLICATION.

PRELIMINARY IDEAS.

486. In algebra, **multiplication** is often indicated only, and the final answer is frequently nothing more than the symbols so written that it is shown they should be multiplied when their numerical values are substituted. Thus the product of m and n is mn , the absence of a sign between the two quantities denoting the operation of multiplication. If we multiply $2a$ by $3b$, the four factors which form the product are 2, a , 3, and b ; hence, $2a \times 3b = 2 \times a \times 3 \times b$. Since we are at liberty to arrange the factors in any order, we have

$$2a \times 3b = 2 \times 3 \times a \times b = 6 \times a \times b = 6ab.$$

Hence, in finding the product of two quantities, *the coefficients are multiplied together and prefixed to the literal factors.*

487. When two quantities are multiplied together, *the sign of the product is positive if the two quantities have the*

same sign; and it is negative if the quantities have opposite signs.

Thus,

$$(+5n) \times (+3m) = +15mn.$$

$$(-5n) \times (+3m) = -15mn.$$

$$(-5n) \times (-3m) = +15mn.$$

$$(+5n) \times (-3m) = -15mn.$$

488. Assume that the $3m$ above is the multiplier, and that $5n$ is the multiplicand. The multiplier must be considered as an abstract number, and simply shows how many times the multiplicand is to be taken; thus, $3m$ may represent 15, 21, 30, etc., depending upon the value of m . The multiplicand, $5n$, may represent any concrete number; as, \$20, 60 feet, 100 miles. The sign of the multiplicand shows the character of the quantity; thus, if $+5n$ represents \$20, $-5n$ may be taken to represent a debt of \$20. The sign of the multiplier, on the other hand, is a symbol of operation and shows how the result is to be treated after it is obtained. If the multiplier is positive, the result obtained is to be added when taken in connection with other quantities; the multiplier is given the negative sign to show that the result is to be subtracted.

Multiplying $+5n$ by $3m$, considering the multiplier simply as an abstract number and disregarding its sign, we obtain $+15mn$. Similarly, $-5n$ multiplied by $3m$ gives $-15mn$. The product in each case has the same character as the multiplicand. Now, if the multiplier has the positive sign, the product is to be added when taken in connection with other quantities, and, therefore, the signs remain as above. That is, $(+5n) \times (+3m) = +15mn$, and $(-5n) \times (+3m) = -15mn$.

If, on the other hand, the multiplier has the negative sign, the product is to be subtracted, and, therefore, its sign must be changed when combined with other quantities. Thus, when the multiplier, $3m$, has the negative sign, we have $(+5n) \times (-3m) = -15mn$ and $(-5n) \times (-3m) = +15mn$.

489. Exponents.—It has been shown that $a^2 = a \times a$ and $a^3 = a \times a \times a$. The product $a^2 \times a^3$, or $a^2 a^3 =$

$a \times a \times a \times a \times a = a^5$. The exponent 2 shows that a is used twice as a factor, and the exponent 3 shows that a is used three times as a factor. In the product, a must be used five times as a factor, or the exponent of a is 5. *The exponent of the product is the sum of the exponents of the factors.*

Thus,

$$a^4 \times a^3 = a^{4+3} = a^7.$$

$$c^2 \times c^4 = c^{2+4} = c^6.$$

$$m \times m^2 \times m^3 = m^{1+2+3} = m^6.$$

$$n^2 \times n^4 \times n^5 \times n^3 = n^{14}.$$

MULTIPLICATION OF MONOMIALS.

490. Rule.—*To the product of the coefficients annex the letters of both factors; give each letter an exponent equal to the sum of the exponents of that letter.*

Make the sign of the product plus, when the signs of the factors are alike; and minus, when they are unlike.

EXAMPLE.—Multiply $4a^2b$ by $-5a^3bc$.

SOLUTION.—The product of the coefficients is 20, and the letters to be annexed are a , b , and c . The new exponent of a is 5, and of b , 2, since $a^2 + 3 = a^5$, and $b^1 + 1 = b^2$. The sign of the product is minus, since the two factors have different signs. Hence, $4a^2b \times -5a^3bc = -20a^5b^2c$. Ans.

491. When there are more than two factors, we have simply three or more examples in multiplication to solve in succession, each to be performed by the foregoing rule.

EXAMPLE.—Find the continued product of $6x^2yz^3$, $-9x^2y^2z^2$, and $-3x^4yz$.

SOLUTION.— $6x^2yz^3 \times -9x^2y^2z^2 = -54x^{2+2}y^{1+2}z^{3+2}$, or $-54x^4y^3z^5$. Now, multiplying this product by $-3x^4yz$, we have $-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6$. Ans.

EXAMPLES FOR PRACTICE.

492. Find the product of:

1. a^2b^3 and $-5abd$.

Ans. $-5a^4b^3d$.

2. $-7xy$ and $-7x^2y^2$.

Ans. $49x^3y^3$.

3. $-15m^5n^6$ and $3mn$.

Ans. $-45m^6n^7$.

4. $3a(x-y)^2$ and $2a^2(x-y)$.

Ans. $6a^3(x-y)^3$.

SUGGESTION.—Treat the $(x-y)$ as though it were a single letter.

5. Find the continued product of $2a^2m^2x$, $-3a^2mx^3$, and $4am^3x^2$.
 Ans. $-24a^4m^4x^4$.
6. What does $-a^3bn \times -2cdn \times -3bdc^2 \times -2acn^2$ equal?
 Ans. $12a^3b^2c^2d^2n^4$.

MULTIPLICATION OF POLYNOMIALS.

493. When one of the factors is a monomial:

Rule.—*Multiply each term of the polynomial by the monomial, and connect the separate products by their proper signs.*

EXAMPLE.—Find the product of $-9a^5 + 3a^3b^2 - 4a^2b^3 - b^5$ and $-3ab^4$.

$$\begin{array}{r} \text{SOLUTION.} \quad -9a^5 + 3a^3b^2 - 4a^2b^3 - b^5 \\ \quad \quad \quad -3ab^4 \\ \hline 27a^6b^4 - 9a^4b^6 + 12a^3b^7 + 3ab^9. \quad \text{Ans.} \end{array}$$

494. When both factors are polynomials:

Rule.—*Multiply each term of one polynomial by each term of the other, and add the partial products.*

EXAMPLE.—Multiply $6a - 4b$ by $4a - 2b$.

SOLUTION.—Write the multiplier under the multiplicand, and begin to multiply *at the left* instead of at the right, as in arithmetic, since polynomials are always written and read from the left, and there are no numbers to carry.

$$\begin{array}{r} 6a - 4b \quad (1) \\ 4a - 2b \\ \hline \text{Multiplying (1) by } 4a \text{ gives } 24a^2 - 16ab \quad (2) \\ \text{Multiplying (1) by } -2b \text{ gives } -12ab + 8b^2 \quad (3) \\ \text{Adding (2) and (3) gives } 24a^2 - 28ab + 8b^2. \quad \text{Ans.} \end{array}$$

It will be noticed that the like terms, $-16ab$ and $-12ab$, are written under each other, so that it will be easier to add them.

EXAMPLE.—Multiply $x^3 - x + 1 + x^2$ by $1 - x^2 + x$.

SOLUTION.—With a view to bringing like terms in the same columns, arrange both multiplicand and multiplier either according to the increasing or the decreasing powers of the same letter. (Art. 455.) Arranging in this case according to the increasing powers of x , we have

$$\begin{array}{r} 1 - x + x^2 + x^3 \quad (1) \\ 1 + x - x^2 \\ \hline \text{Multiplying (1) by 1 gives } 1 - x + x^2 + x^3 \quad (2) \\ \text{Multiplying (1) by } +x \text{ gives } x - x^2 + x^3 + x^4 \quad (3) \\ \text{Multiplying (1) by } -x^2 \text{ gives } -x^2 + x^3 - x^4 - x^5 \quad (4) \\ \text{Adding (2), (3), and (4) gives } 1 - x^2 + 3x^3 - x^5. \quad \text{Ans.} \end{array}$$

495. Multiplication is frequently indicated by enclosing each of the quantities to be multiplied in a parenthesis. The sign of multiplication is not placed between the parentheses, multiplication being understood. When the quantities are multiplied together, the expression is said to be *expanded*.

For example, in the expression $(m - 2n)(2m - n)$, the binomial $m - 2n$ is to be multiplied by the binomial $2m - n$. Performing the multiplication, the product is $2m^2 - 5mn + 2n^2$, which is the expanded form of the expression.

EXAMPLES FOR PRACTICE.

496. Multiply the following:

1. $x^3 + 2xy + y^2$ by $x + y$. Ans. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $3ab^2m^3 + 4a^2b - 2$ by $a^6b^7m^8$. Ans. $3a^7b^9m^{11} + 4a^8b^9m^8 - 2a^6b^7m^8$.
3. $c^3 - d^3$ by $c^2 + d^2$. Ans. $c^5 - d^5$.
4. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$. Ans. $x^6 - y^6$.
5. $3a^2 - 7a + 4$ by $2a^3 + 9a - 5$. Ans. $6a^5 + 13a^3 - 70a^2 + 71a - 20$.

Expand the following:

6. $(2a - 3c)(4 - 3a)$. Ans. $8a - 12c - 6a^2 + 9ac$.
7. $(x + 2)(x - 2)(x^2 + 4)$. Ans. $x^4 - 16$.
8. $[x(x^2 - y^2) - 2][x(x^2 + y^2) + 2]$.

NOTE.—The expressions in the brackets reduce to $x^3 - xy^2 - 2$ and $x^3 + xy^2 + 2$. The product of these is $x^6 - x^2y^4 - 4xy^2 - 4$. Ans.

THREE IMPORTANT EXAMPLES.

497. Let a and b be *any* two quantities; we wish to find the forms of the following products:

$$(a + b)^2, (a - b)^2, \text{ and } (a + b)(a - b).$$

By actual multiplication we find

$$(a + b)^2 = a^2 + 2ab + b^2. \quad (1.)$$

$$(a - b)^2 = a^2 - 2ab + b^2. \quad (2.)$$

$$(a + b)(a - b) = a^2 - b^2. \quad (3.)$$

Hence:

498. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and the second, plus the square of the second.*

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and the second, plus the square of the second.

The product of the sum and difference of two quantities is equal to the difference of their squares.

499. The foregoing statements should be committed to memory, since their use will frequently save tedious calculation. *The student is advised to practice until he is certain that he knows them perfectly.*

EXAMPLE.—Square $3x^2 + 5$.

SOLUTION.—The square of the first term is $3x^2 \times 3x^2 = 9x^4$; twice the product of the terms is $30x^2$; and the square of the last term is 25. Hence, by formula 1, letting $a = 3x^2$ and $b = 5$,

$$(3x^2 + 5)^2 = 9x^4 + 30x^2 + 25. \quad \text{Ans.}$$

EXAMPLE.—Square $4cd - x$.

SOLUTION.—The square of the first term is $16c^2d^2$; twice the product of the first and the second is $8cdx$; and the square of the last term is x^2 . Hence, by formula 2, letting $a = 4cd$ and $b = x$,

$$(4cd - x)^2 = 16c^2d^2 - 8cdx + x^2. \quad \text{Ans.}$$

EXAMPLE.—Expand $(x^2 + 3)(x^2 - 3)$. (See Art. 495.)

SOLUTION.—The square of the first term is x^4 , and of the second, 9. Hence, by formula 3, letting $a = x^2$ and $b = 3$,

$$(x^2 + 3)(x^2 - 3) = x^4 - 9. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

500. Square the following:

1. $m + n$.

Ans. $m^2 + 2mn + n^2$.

2. $4x + 2$.

Ans. $16x^2 + 16x + 4$.

3. $3a - 5b$.

Ans. $9a^2 - 30ab + 25b^2$.

Expand the following:

4. $(m + 1)(m - 1)$.

Ans. $m^2 - 1$.

5. $(x^2 + y^2)(x^2 - y^2)$.

Ans. $x^4 - y^4$.

6. $(4a + 4b^2)(4a - 4b^2).$

Ans. $16a^2 - 16b^4.$

7. Square $2c^2 - c + d.$

NOTE.—First separate $2c^2 - c + d$ into two terms by enclosing $c + d$ in parenthesis; then the expression becomes $2c^2 - (c - d)$, and considering this as a binomial we find the square to be $4c^4 - 4c^2(c - d) + (c - d)^2.$

$$\begin{aligned} -4c^2(c - d) &= -4c^3 + 4c^2d, \\ (c - d)^2 &= c^2 - 2cd + d^2. \end{aligned}$$

Adding these results to $4c^4$, the final result is $4c^4 - 4c^3 + 4c^2d + c^2 - 2cd + d^2.$ Ans.

DIVISION.

INTRODUCTORY.

501. When two quantities are given, and we wish to find a third quantity which, if multiplied by one of the first two, will produce the other, the process of finding this third quantity is called **division**. Thus, if the given quantities are ab and a , and we wish to find a quantity which, if multiplied by a , will give ab , we must divide ab by a ; our quotient will be b , since $a \times b = ab$. Division is, therefore, the inverse of multiplication.

502. The following laws of division follow directly from the statements of Arts. **486** to **489**:

If the dividend and the divisor have like signs, the quotient will have the plus sign; if they have unlike signs, the quotient will have the minus sign.

The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.

The exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

503. Let it be required to divide a^3 by a^2 . We have to obtain a quotient, which, when multiplied by the divisor a^2 , will produce the dividend a^3 . The quotient is evidently 1. By Art. **502**, however, we know that $a^3 \div a^2 = a^{3-2} = a^1$. Hence, *any quantity whose exponent is 0 is equal to 1.*

From the foregoing principles, the rules for division are obtained.

DIVISION OF MONOMIALS.

504. Rule.—Divide the coefficient of the dividend by the coefficient of the divisor and to the quotient annex the letters of the dividend, each with an exponent equal to its exponent in the dividend minus its exponent in the divisor, omitting those letters whose exponents become zero.

Make the sign of the quotient plus when the dividend and divisor have like signs, and minus when they have unlike signs.

EXAMPLE.—Divide $6a^5b^4c^3$ by $-3a^3bc^3$.

SOLUTION.—The quotient of $6 \div 3$ is 2. The letters to be annexed, and their exponents, are $a^{5-3} = a^2$, and $b^{4-1} = b^3$. The c has an exponent of $3 - 3 = 0$, so that it becomes equal to 1, and is omitted. The sign of the quotient is minus. Hence, $6a^5b^4c^3 \div -3a^3bc^3 = -2a^2b^3$. Ans.

PROOF.— $-3a^3bc^3 \times -2a^2b^3 = 6a^5b^4c^3$.

EXAMPLE.—Divide $-10a^6b^3c^2d$ by $-2ab^3c$.

SOLUTION.— $-10a^6b^3c^2d \div -2ab^3c = 5a^{6-1}b^{3-3}c^{2-1}d = 5a^5cd$. Ans.

EXAMPLES FOR PRACTICE.

505. Divide the following:

1. $12m^2n$ by $4n$. Ans. $3m^2$.
2. $30x^6y^5bc^3$ by $-6x^5y^5c^2$. Ans. $-5xbc$.
3. $-44a^3b^2c^3$ by $-11ab^2c^2$. Ans. $4a^2b$.
4. $-100x^4y^3z^2$ by x^3y^2 . Ans. $-100xyz^2$.
5. $75pq^2x^2m^4$ by $75x^2$. Ans. pq^2m^4 .

DIVISION OF POLYNOMIALS.

506. When the divisor is a monomial:

Rule.—Divide each term of the dividend by the divisor, and connect the partial quotients by their proper signs.

EXAMPLE.—Divide $12a^2b^4 - 9ab^3 + 6a^3b^4$ by $3ab^3$.

SOLUTION.—
$$\begin{array}{r} 3ab^3 \overline{) 12a^2b^4 - 9ab^3 + 6a^3b^4} \\ \underline{12a^2b^4 - 9ab^3} \\ 0 \end{array}$$

 quotient $4ab - 3 + 2a^2b$. Ans.

EXAMPLES FOR PRACTICE.

507. Divide the following:

1. $64m^2n^3 - 32mn^3 + 8m^2n$ by $8mn$. Ans. $8mn^2 - 4n + m$.

2. $27x^3yz - 9x^3yz^2 - 333x^3yz^2$ by $-3x^3yz$.
Ans. $-9y + 3z + 111yz$.

3. $10(x+y)^3 - 5a(x+y) + 5a^3(x+y)$ by $5(x+y)$.
Ans. $2(x+y) - a + a^3$.

508. When the divisor is a polynomial:**Rule.**—Arrange both dividend and divisor according to the ascending or descending powers of some letter.

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Subtract from the dividend the product of the divisor and this term of the quotient.

Treat the remainder as a new dividend, and proceed as before, until there is no remainder, or until the final remainder contains no term which is divisible by the first term of the divisor.

EXAMPLE.—Divide $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.**SOLUTION.**—

$$\begin{array}{r}
 x^4 + 4x^3 + 4x^2 \quad \text{quotient.} \\
 x^4 + 4x^3 + 4x^2 \quad \text{Ans.} \\
 \hline
 -3x^2 - 13x^2 - 16x \\
 -3x^2 - 12x^2 - 12x \\
 \hline
 -x^2 - 4x - 4 \\
 -x^2 - 4x - 4 \\
 \hline
 0 \quad 0 \quad 0
 \end{array}$$

The first term x^2 of the divisor is contained in x^4 , the first term of the dividend, x^2 times; hence, x^2 is the first term of the quotient. The whole divisor multiplied by this term gives $x^4 + 4x^3 + 4x^2$ as a product, which subtracted from the dividend gives as a remainder, $-3x^2 - 13x^2 - 16x - 4$. It is not necessary here to bring down the -4 , since only three terms are required to contain the divisor.

The first term x^2 of the divisor is contained in $-3x^2$, the first term of the new dividend, $-3x$ times. Multiplying the divisor by this new term of the quotient, we have $-3x^3$

$-12x^3 - 12x$. Subtracting this from the first remainder, we obtain $-x^3 - 4x - 4$ for a new remainder, the -4 being brought down from the original dividend. The first term of the divisor is contained in the first term of the new remainder or dividend, -1 times. Multiplying the divisor by this, we get $-x^3 - 4x - 4$, which subtracted from $-x^3 - 4x - 4$, the last remainder, leaves a difference of zero. The work ends here, since there are no more terms in the dividend to be brought down.

EXAMPLE.—Divide $9x^3y^3 + x^4 - 4y^4 - 6x^2y$ by $x^2 + 2y^2 - 3xy$.

SOLUTION.—First arrange the dividend and divisor according to the descending powers of x .

$$\begin{array}{r}
 x^2 - 3xy + 2y^2 \overline{) x^4 - 6x^2y + 9x^2y^3 - 4y^4} \quad (x^2 - 3xy - 2y^2 \quad \text{Ans.} \\
 \underline{x^4 - 3x^2y + 2x^2y^3} \\
 - 3x^2y + 7x^2y^3 \qquad - 4y^4 \\
 \underline{- 3x^2y + 9x^2y^3 - 6xy^3} \\
 - 2x^2y^3 + 6xy^3 - 4y^4 \\
 \underline{- 2x^2y^3 + 6xy^3 - 4y^4} \\
 0 \qquad 0 \qquad 0
 \end{array}$$

EXAMPLES FOR PRACTICE.

509. Divide the following:

1. $x^3 - 7x + 12$ by $x - 3$. Ans. $x - 4$.
2. $x^3 + x - 72$ by $x + 9$. Ans. $x - 8$.
3. $2x^3 - x^2 + 3x - 9$ by $2x - 3$. Ans. $x^2 + x + 3$.
4. $x^4 + 11x^3 - 12x - 5x^2 + 6$ by $3 + x^2 - 3x$. Ans. $x^2 - 2x + 2$.
5. $x^4 - 6xy - 9x^2 - y^3$ by $x^2 + y + 3x$. Ans. $x^2 - 3x - y$.
6. $x^6 - 1$ by $x - 1$. Ans. $x^5 + x^4 + x^3 + x^2 + x + 1$

FACTORING.

510. Factoring is the process of finding the *factors* of a quantity, that is, the quantities or numbers which will divide that quantity without a remainder.

511. Expanding $6a^2b(2b + a)$, we have $12a^2b^2 + 6a^3b$; hence, $6a^2b$ and $(2b + a)$ are the factors of $12a^2b^2 + 6a^3b$.

The monomial factor $6a^3b$ may be further resolved into $3 \times 2 \times a \times a \times b$. In solving examples in factoring, it is not customary to write out the factors of a monomial, since they are generally apparent.

That the student may be able to recognize factors without the labor of actual division, several methods of readily discovering factors are here given.

512. Equal factors are those whose terms have the same letters, and whose letters have the same exponents and the same signs. Thus, $5a(2y - x)$ and $5a(2y - x)$ are equal factors of $5a(2y - x) \times 5a(2y - x) = 25a^2(2y - x)^2$; but $5a(2y - x)$ and $-5a(2y - x)$ are unequal factors, since the signs of $5a$ are not the same in both expressions.

513. A product of two equal factors is a **perfect square**. Either of the equal factors of a quantity is called its square root.

514. A product of three equal factors is a **perfect cube**. Any one of the equal factors of a quantity is called its cube root.

515. In factoring, it is important to be able to easily distinguish quantities that are perfect squares and cubes, and to determine their roots. By definition, $9a^2b^2$ is a perfect square because $3ab \times 3ab = 9a^2b^2$, and $3ab$ is its square root. Also, $8a^3$ is a perfect cube because $2a \times 2a \times 2a = 8a^3$, and $2a$ is its cube root. In each of these cases the coefficients of the roots are multiplied together, and the exponents added, to produce a perfect power. *Hence, a quantity is a perfect square when its coefficient is a perfect square, and the exponents of all its letters can be divided by 2.* For example, $36x^{10}$, $49b^2c^4d^8$, $16a^6b^{12}$, and 1 are all perfect squares, whose roots are $6x^5$, $7bc^2d^4$, $4a^3b^6$, and 1, respectively. No perfect square, however, can have a minus sign; for, let a = any quantity, $-a \times -a = a^2$, and $a \times a = a^2$. The square root of a^2 may be $-a$, or a , and a square root is often written $\pm a$, read plus or minus a .

A quantity is a perfect cube when its coefficient is a perfect cube, and the exponents of all its letters can be divided by 3. Thus, $27x^{12}$, $-64b^3c^3d^3$, $8a^{12}b^{12}$, and 1 are all perfect cubes, whose roots are $3x^4$, $-4bc^3d^3$, $2a^4b^4$, and 1, respectively. The sign of the cube root is always the same as that of its cube.

CASE I.

516. *When all the terms of an expression are divisible by the same quantity, the expression may be resolved into two factors by dividing it by that quantity.*

EXAMPLE.—Factor $8m^4n^3 - 10m^3n^2y + 2m^2n^3$.

SOLUTION.—We first examine the polynomial to see if one of its terms is contained in each of the others. Beginning with the smallest coefficient, 2, we find it to be contained in each of the others. We next take the literal portion m^2n^3 of this term, and find both of its letters in each of the other terms, with an equal or higher exponent. Dividing through by $2m^2n^3$, we have for our quotient $4mn - 5my + 1$. Hence, we have

$$8m^4n^3 - 10m^3n^2y + 2m^2n^3 = 2m^2n^3(4mn - 5my + 1). \quad \text{Ans.}$$

EXAMPLE.—Ascertain if $12ab^3c^3 - 18a^3c^3y + 24a^2c^4 - 36a^2bc^3y^2$ has a monomial factor.

SOLUTION.—By inspection we find that the smallest coefficient is not contained in the other three without a remainder. This coefficient, 12, is the product of $3 \times 2 \times 2$. Of these, both 2 and 3 will divide all the coefficients once. Therefore, $2 \times 3 = 6$ is the largest numerical factor. The letters a and c are contained in all the terms; the smallest exponent of a is 1 and of c is 2. The monomial factor is evidently $6ac^2$. Dividing the polynomial by $6ac^2$ the quotient is $2b^3c - 3a^2y + 4ac^2 - 6abc^3y^2$; the factors are $6ac^2$ and $2b^3c - 3a^2y + 4ac^2 - 6abc^3y^2$. Ans.

EXAMPLES FOR PRACTICE.

517. Factor the following expressions:

- | | |
|--|-------------------------------------|
| 1. $a^4 + ax.$ | Ans. $a(a^3 + x).$ |
| 2. $12a^5 - 2a^3 + 4a^4.$ | Ans. $2a^3(6a^2 - 1 + 2a).$ |
| 3. $30m^4n^3 - 6n^4.$ | Ans. $6n^3(5m^4 - n).$ |
| 4. $16x^3y^3 - 8x^5 + 8.$ | Ans. $8(2x^3y^3 - x^5 + 1).$ |
| 5. $4x^2y - 12x^2y^2 + 8xy^3.$ | Ans. $4xy(x^2 - 3xy + 2y^2).$ |
| 6. $49a^3b^3c^4 - 63a^3b^3c^4 + 7a^4b^3c^3.$ | Ans. $7a^3b^3c^3(7bc - 9ac + a^2).$ |

CASE II.

518. *To factor a trinomial which is a perfect square:*

Any trinomial is a perfect square when the first and the last term are perfect squares and positive, and the second term is twice the product of their square roots.

Thus, let a and b represent any two quantities whatever, and we have the general forms of the square as follows:

$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2. \quad (4.)$$

$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2. \quad (5.)$$

These, it will be seen, are simply the inverse of formulas **1** and **2**, Art. **497**. The sign of the second term of the square always determines the sign of the second term of the root, b in this particular case.

519. Since a may represent one quantity and b any other quantity, it is evident that *any* trinomial having the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect square.

Rule.—*Extract the square roots of the first and the last term of the trinomial, and connect the results by the sign of the second term.*

EXAMPLE.—Factor $x^2 + 2xy + y^2$.

SOLUTION.—We first see if the trinomial has the form stated in Art. **518**. The first and the last term we see to be perfect squares, and their roots to be x and y . The second term is also twice the product of the roots x and y , and, since it has the plus sign, the binomial root must be $x + y$. Hence, we have a square of the form $a^2 + 2ab + b^2$, and

$$x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2. \quad \text{Ans.}$$

EXAMPLE.—Factor $16m^4 + 9n^6 - 24m^2n^3$.

SOLUTION.—The first term of the expression is a perfect square, but the last term is not. Inspecting the second term we find it to be the square of $3n^3$, and the third term to be twice the product of $3n^3$ and the square root, $4m^2$, of the first term. Arranging the trinomial so that the first and the last term are perfect squares, we get $16m^4 - 24m^2n^3 + 9n^6$ (a square of the form $a^2 - 2ab + b^2$), and we have $16m^4 + 9n^6 - 24m^2n^3 = 16m^4 - 24m^2n^3 + 9n^6 = (4m^2 - 3n^3)(4m^2 - 3n^3) = (4m^2 - 3n^3)^2$. Ans.

EXAMPLE.—Factor $4x^2 + x^2y^2 + 2x^2y$.

SOLUTION.—Arranging the trinomial so that the first and the last term are perfect squares, we have $4x^2 + 2x^2y + x^2y^2$. Now, although the first and the last term are perfect squares with roots $2x$ and xy , respectively, the second term is only equal to the product of the roots; hence, the trinomial is *not* a perfect square, and can only be factored by Case I. Each term contains x^2 , and we have $4x^2 + x^2y^2 + 2x^2y = x^2(4 + y^2 + 2y)$. Ans.

520. *Should two of the terms of a trinomial be perfect squares, and have like signs, and the other term be twice the product of their roots, the trinomial is a perfect square.*

Compare this statement with Art. 518. Thus, $2ab - a^2 - b^2$, if divided by -1 , becomes $-2ab + a^2 + b^2 = a^2 - 2ab + b^2$; hence, $2ab - a^2 - b^2 = -(a^2 - 2ab + b^2) = -(a - b)^2$.

EXAMPLE.—Factor $4pq - 4p^2 - q^2$.

SOLUTION.—Dividing first by -1 we have $-4pq + 4p^2 + q^2 = 4p^2 - 4pq + q^2 = (2p - q)^2$. Hence, $4pq - 4p^2 - q^2 = -(4p^2 - 4pq + q^2) = -(2p - q)^2$. Ans.

EXAMPLE.—Factor $16r^2s^2 + 16r^4 + 4s^4$.

SOLUTION.—The expression contains three squares, but, by careful inspection, we see that the first term is also twice the product of the square roots of the other two. Thus, $16r^2s^2 + 16r^4 + 4s^4 = 16r^4 + 16r^2s^2 + 4s^4 = (4r^2 + 2s^2)^2$. Ans.

EXAMPLES FOR PRACTICE.

521. Factor the following trinomials:

- | | |
|---------------------------------|--------------------------|
| 1. $x^2 - 16x + 64$. | Ans. $(x - 8)^2$. |
| 2. $n^2 - 26n^2 + 169$. | Ans. $(n^2 - 13)^2$. |
| 3. $25x^2 + 70xyz + 49y^2z^2$. | Ans. $(5x + 7yz)^2$. |
| 4. $16c^2 + b^2 - 8bc$. | Ans. $(4c - b)^2$. |
| 5. $2mx - m^2 - x^2$. | Ans. $-(m - x)^2$. |
| 6. $a^2b^4c^4 - 2ab^2c^3 + 1$. | Ans. $(ab^2c^3 - 1)^2$. |

CASE III.

522. *To factor an expression which is the difference between two perfect squares:*

This case is the inverse of formula 3, Art. 497, and may be expressed by the formula

$$a^2 - b^2 = (a + b)(a - b). \quad (6.)$$

523. Since a may represent one quantity and b any other quantity, it is evident from formula **6** that any expression which is the difference between two perfect squares may be factored by the following

Rule.—*Extract the square roots of the first and the last term. Write the first root plus the second for one factor, and the first root minus the second for the other.*

EXAMPLE.—Factor $9x^2y^4 - 4$.

SOLUTION.—The square roots of the first and the last term are $3x^2y^2$ and 2. The sum of these roots is $3x^2y^2 + 2$, and the second subtracted from the first is $3x^2y^2 - 2$. Hence, by formula **6**, letting $a = 3x^2y^2$ and $b = 2$,

$$9x^2y^4 - 4 = (3x^2y^2 + 2)(3x^2y^2 - 2). \quad \text{Ans.}$$

EXAMPLE.—Factor $(a + b)^2 - m^2n^2$.

SOLUTION.—The square roots of the first and the last term are $a + b$ and mn . The sum of these roots is $a + b + mn$, and the second subtracted from the first is $a + b - mn$. Hence, by formula **6**, letting $a = a + b$ and $b = mn$,

$$(a + b)^2 - m^2n^2 = (a + b + mn)(a + b - mn). \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

524. Factor the following expressions:

1. $a^2 - 16$. Ans. $(a + 4)(a - 4)$.

2. $a^2 - 49c^2$. Ans. $(a + 7c)(a - 7c)$.

3. $81x^2y^4 - 1$. Ans. $(9x^2y^2 + 1)(9x^2y^2 - 1)$.

4. $(ax + by)^2 - 1$. Ans. $(ax + by + 1)(ax + by - 1)$.

5. $25x^2y^2 - (bx + 1)^2$. Ans. $[5x^2y + (bx + 1)][5x^2y - (bx + 1)] =$
 $(5x^2y + bx + 1)(5x^2y - bx - 1)$.

6. $1 - 169x^2y^4z^6$. Ans. $(1 + 13xy^2z^3)(1 - 13xy^2z^3)$.

525. In example 5, the expression $(bx + 1)^2$ should be regarded as a single term; in fact, any number of terms may be regarded as a single term by enclosing them in parenthesis and operating on them as though they were a single letter.

526. When solving any examples requiring the application of the rule in Art. **523**, first ascertain if the numerical coefficients of the two terms are perfect squares; if not, there is no use of examining further.

FRACTIONS.

DEFINITIONS.

527. A **fraction**, in algebra, is considered as an expression indicating division. The sign $\div$ is seldom used, it being more convenient to write the dividend, or quantity to be divided, above a horizontal line, with the divisor below it, in the form of a fraction.

Thus the fraction $\frac{a+b}{c-d}$ means that $a+b$ is to be divided by $c-d$, and is the same as $(a+b) \div (c-d)$. It is read " $a+b$ divided by $c-d$ " or " $a+b$ over $c-d$." All fractions are read in this way in algebra, except simple numerical fractions, as $\frac{1}{2}$, $\frac{3}{4}$, etc., which are read as in arithmetic.

528. The quantities above and below the line are called the **numerator** and the **denominator**, respectively, as in the case of numerical fractions. They are known as the **terms** of a fraction.

529. Since dividing any quantity by 1 does not change its value, we may write any quantity as a fraction by making the quantity itself the numerator, and 1 the denominator. Thus, $7x^2y$ may be written $\frac{7x^2y}{1}$, and not be altered in value.

530. A **reciprocal** of a quantity is 1 divided by that quantity. It is not necessarily written as a fraction, but when so written has the quantity for the denominator, and 1 for the numerator. Thus, the reciprocal of 5 is $\frac{1}{5}$, but the reciprocal of $\frac{1}{5}$ is $\frac{5}{1} = 5$; the reciprocal of $x^2 + y^2 = \frac{1}{x^2 + y^2}$; and the reciprocal of $\frac{2}{x^2 + y^2} = \frac{x^2 + y^2}{2}$. Hence, the reciprocal of a fraction may be obtained by inverting the fraction.

531. The **three signs of a fraction** are: the sign before the dividing line, which affects the entire fraction; the sign of the numerator; and the sign of the denominator.

When any one of these signs is omitted, it is understood to be plus. *Any two signs of a fraction may be changed without altering its value, but if any one, or all three, be changed, the value of the fraction will be changed from + to - or from - to +.*

When either the numerator or the denominator has more than one term, it should be enclosed in a parenthesis when performing operations affecting it as a whole. The parenthesis may be removed after the operations are completed.

Take the fraction $-\frac{a-b}{c-d}$; placing numerator and denominator in parentheses, we have $-\frac{(a-b)}{(c-d)}$. The signs of the numerator and denominator are each + and that of the fraction -.

Let the quotient of $a-b \div c-d = q$; then:

$$-\frac{+(a-b)}{+(c-d)} = -(+q) = -q.$$

$$-\frac{-(a-b)}{-(c-d)} = -(+q) = -q.$$

$$-\frac{+(a-b)}{-(c-d)} = -(-q) = +q.$$

$$-\frac{-(a-b)}{+(c-d)} = -(-q) = +q.$$

$$+\frac{+(a-b)}{+(c-d)} = +(+q) = +q.$$

$$+\frac{-(a-b)}{-(c-d)} = +(+q) = +q.$$

$$+\frac{+(a-b)}{-(c-d)} = +(-q) = -q.$$

$$+\frac{-(a-b)}{+(c-d)} = +(-q) = -q.$$

REDUCTION OF FRACTIONS.

532. To *reduce* a fraction is to change its form without changing its value. Thus, $\frac{10x}{5}$ and $\frac{20x}{10}$ have different forms, but like values, since $10x \div 5$ and $20x \div 10$ are each equal to $2x$.

The terms of a fraction may both be multiplied, or may both be divided by the same quantity without changing their value.

533. To reduce a fraction to its simplest form:

Rule.—Resolve each term into its factors, and cancel factors which appear in both.

534. In performing all operations on fractions, the student must learn to use a polynomial factor as a single quantity, like a monomial factor.

This is illustrated in the following examples, where there are polynomial factors in both numerator and denominator that can be canceled.

EXAMPLE.—Reduce $\frac{x^2 + 2xy + y^2}{x^2 - y^2}$ to its simplest form.

SOLUTION.—Factoring both numerator and denominator,

$$\frac{x^2 + 2xy + y^2}{x^2 - y^2} = \frac{(x+y)(x+y)}{(x+y)(x-y)}$$

Canceling the common factor $x+y$ from both gives, as the result,

$$\frac{\cancel{(x+y)}(x+y)}{\cancel{(x+y)}(x-y)} = \frac{x+y}{x-y}. \quad \text{Ans}$$

EXAMPLE.—Reduce $\frac{3x^5 - 6x^4y}{6x^3y^2 - 12xy^3}$ to its simplest form.

SOLUTION.— $\frac{3x^5 - 6x^4y}{6x^3y^2 - 12xy^3} = \frac{3x^4(x-2y)}{6xy^2(x-2y)}$, when factored.

Canceling the common factors, we have as the result,

$$\frac{\cancel{3x^4}(x-2y)}{\cancel{6xy^2}(x-2y)} = \frac{x^3}{2y^2}. \quad \text{Ans}$$

535. Sometimes the whole numerator is contained in the denominator, or the denominator in the numerator. The numerator or denominator will then reduce to the number 1.

EXAMPLE.—Reduce $\frac{b+3c^2}{2b^2+6bc^2}$ to its simplest form.

SOLUTION.— $\frac{b+3c^2}{2b^2+6bc^2} = \frac{\cancel{b+3c^2}}{2b(\cancel{b+3c^2})} = \frac{1}{2b}. \quad \text{Ans}$

EXAMPLE.—Reduce $\frac{x^6-1}{x^3-1}$ to its simplest form.

SOLUTION.— $\frac{x^6-1}{x^3-1} = \frac{(x^3+1)(\cancel{x^3-1})}{\cancel{x^3-1}} = \frac{x^3+1}{1} = x^3+1$. **Ans.**
(Art. 529.)

536. From the last example it will be seen that division may sometimes be performed by cancelation. Thus, $\frac{x^6-1}{x^3-1}$ means $(x^6-1) \div (x^3-1)$, and the divisor x^3-1 canceled from the dividend x^6-1 gives the quotient x^3+1 .

A factor must be common to each term of the numerator and to each term of the denominator in order to be canceled.

Thus, the factor x can not be canceled from $\frac{3ax}{x+4m}$ because it is not common to both terms of the denominator.

EXAMPLES FOR PRACTICE.

537. Reduce the following to their simplest form:

- | | |
|--------------------------------------|---------------------------|
| 1. $\frac{3a+3b}{a^2-b^2}$. | Ans. $\frac{3}{a-b}$. |
| 2. $\frac{x^4-y^4}{x^2-y^2}$. | Ans. x^2+y^2 . |
| 3. $\frac{54a^3b^5c^3}{72a^2b^3c}$. | Ans. $\frac{3ab^2c}{4}$. |
| 4. $\frac{12a^2x^3}{36a^3x^5}$. | Ans. $\frac{1}{3ax^2}$. |
| 5. $\frac{n^3-2n^2}{n^2-4n+4}$. | Ans. $\frac{n^2}{n-2}$. |

538. When fractions are to be added or subtracted, it is necessary to so reduce them that all the denominators will be alike. This is called **reducing them to a common denominator**. See Arithmetic, Arts. 95 to 99.

539. To reduce fractions to a common denominator:

Rule.—Resolve each denominator into its factors.

Take each factor as many times as it occurs in any one denominator, and find the product of these factors.

Divide this product by each of the denominators. Multiply the corresponding numerators by these quotients, for new numerators. Write each new numerator with the common denominator beneath it.

EXAMPLE.—Reduce $\frac{7a}{x+y}$, $\frac{3ab}{x^2-y^2}$, and $\frac{2b}{(x+y)^2}$ to a common denominator.

SOLUTION.—Factoring the denominators, we have $x+y$ not factorable. $x^2-y^2=(x+y)(x-y)$, and $(x+y)^2=(x+y)(x+y)$. Now we have two separate factors, $x+y$ and $x-y$, of which $x+y$ occurs twice in $(x+y)^2$. Hence, our common denominator is $(x+y)(x+y)(x-y)=(x+y)^2(x-y)$. Dividing this product by $x+y$ we have $(x+y)(x-y)=x^2-y^2$ as our quotient. Hence, our first new numerator is $7a(x^2-y^2)$ and the new fraction is $\frac{7a(x^2-y^2)}{(x+y)^2(x-y)}$. Similarly, $\frac{3ab}{x^2-y^2}$ becomes $\frac{3ab(x+y)}{(x+y)^2(x-y)}$, and $\frac{2b}{(x+y)^2}$ becomes $\frac{2b(x-y)}{(x+y)^2(x-y)}$.

The student should note that this denominator can be written in several different ways, and he should not become confused if his work does not always agree with the answer. Besides $(x+y)(x+y)(x-y)$ and $(x+y)^2(x-y)$, it may be written $(x^2-y^2)(x+y)$, $(x^2+2xy+y^2)(x-y)$, or $x^3+x^2y-xy^2-y^3$. These five expressions have exactly the same value. The student should prove this statement by substituting numbers for x and y .

EXAMPLES FOR PRACTICE.

540. Reduce the following to a common denominator:

1. $\frac{3yz}{2x}$, $\frac{4xz}{3y}$, and $\frac{5xy}{4z}$. Ans. $\frac{18y^2z^2}{12xyz}$, $\frac{16x^2z^2}{12xyz}$, and $\frac{15x^2y^2}{12xyz}$.
2. $\frac{x^2y}{10}$, $\frac{xyz}{15}$, and $\frac{7yz^2}{30}$. Ans. $\frac{3x^2y}{30}$, $\frac{2xyz}{30}$, and $\frac{7yz^2}{30}$.
3. $\frac{2}{a^3x^3}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$. Ans. $\frac{2}{a^3x^3}$, $\frac{3a^2}{a^3x^3}$, and $\frac{4ax^2}{a^3x^3}$.
4. $\frac{m+n}{m-n}$, and $\frac{m-n}{m+n}$. Ans. $\frac{m^2+2mn+n^2}{m^2-n^2}$, and $\frac{m^2-2mn+n^2}{m^2-n^2}$.
5. $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-1}{4x^2-1}$.
Ans. $\frac{2(4x^2-1)}{x(4x^2-1)}$, $\frac{3x(2x+1)}{x(4x^2-1)}$, and $\frac{x(2x-1)}{x(4x^2-1)}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

541. To add or subtract fractions:

Rule.—Reduce the fractions, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.

EXAMPLE.—Find the sum of $\frac{2a-b}{5}$ and $\frac{a+b}{4}$.

SOLUTION.— $\frac{2a-b}{5}$ and $\frac{a+b}{4}$, reduced to a common denominator, become $\frac{4(2a-b)}{20}$ and $\frac{5(a+b)}{20}$, which are equal, respectively, to $\frac{8a-4b}{20}$ and $\frac{5a+5b}{20}$. Adding the numerators, we have $8a-4b+5a+5b=13a+b$. The result written over the common denominator gives as the sum, $\frac{13a+b}{20}$. The work is written as follows:

$$\begin{aligned}\frac{2a-b}{5} + \frac{a+b}{4} &= \frac{8a-4b}{20} + \frac{5a+5b}{20} = \\ \frac{8a-4b+5a+5b}{20} &= \frac{13a+b}{20}. \quad \text{Ans.}\end{aligned}$$

EXAMPLE.—Subtract $\frac{6b-2}{3b}$ from $\frac{4a-1}{2a}$.

SOLUTION.—Reducing the fractions to a common denominator, $\frac{4a-1}{2a} - \frac{6b-2}{3b} = \frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab}$. Subtracting the second numerator from the first, and writing the result over the common denominator, we have $\frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab} = \frac{(12ab-3b)-(12ab-4a)}{6ab} = \frac{12ab-3b-12ab+4a}{6ab}$, with the parentheses removed. Combining like terms in the numerator gives as the result $\frac{4a-3b}{6ab}$. Ans.

542. If, as in the example just given, the numerator of the fraction to be subtracted has more than one term, care must be taken to change the sign of every term before combining. It will usually be convenient to enclose the whole numerator in a parenthesis before combining. The parenthesis may then be removed by the rules of Arts. 482 and 483.

EXAMPLE.—Simplify $\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1}$.

SOLUTION.—Reducing to the common denominator $x^2 - 1$,

$$\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1} = \frac{x^4 + x^3}{x^2 - 1} - \frac{x^3 - x^2}{x^2 - 1} - \frac{x^2 + x}{x^2 - 1} + \frac{x - 1}{x^2 - 1}.$$

Adding or subtracting the numerators as required, we have

$$\frac{(x^4 + x^3) - (x^3 - x^2) - (x^2 + x) + (x - 1)}{x^2 - 1},$$

which, with the parentheses removed =

$$\frac{x^4 + x^3 - x^3 + x^2 - x^2 - x + x - 1}{x^2 - 1}.$$

Combining like terms we have as the result

$$\frac{x^4 - 1}{x^2 - 1} = x^2 + 1. \quad \text{Ans.}$$

EXAMPLE.—Simplify $\frac{1}{(x-2)^2} + \frac{1}{2-x}$.

SOLUTION.—If the denominator of the second fraction were written $x-2$ instead of $2-x$, $(x-2)^2$ would be the common denominator. By Art. 531, the signs of the denominator and the sign before the fraction $\frac{1}{2-x}$ may be changed, giving $-\frac{1}{-2+x} = -\frac{1}{x-2}$

(Art. 456.) Hence, we have $\frac{1}{(x-2)^2} + \frac{1}{2-x} = \frac{1}{(x-2)^2} - \frac{1}{x-2}$, which, when reduced to a common denominator, is equal to

$$\frac{1}{(x-2)^2} - \frac{x-2}{(x-2)^2} = \frac{1-(x-2)}{(x-2)^2} = \frac{1-x+2}{(x-2)^2} = \frac{3-x}{(x-2)^2}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

543. Simplify the following:

$$1. \quad \frac{x}{3} + \frac{x}{4} + \frac{x}{5}. \quad \text{Ans. } \frac{47x}{60}.$$

$$2. \quad \frac{4x-8}{5} + \frac{7x+1}{3} + \frac{3x}{2}. \quad \text{Ans. } \frac{139x-8}{30}.$$

$$3. \quad \frac{1}{x-y} - \frac{1}{x^2-y^2}. \quad \text{Ans. } \frac{x+y-1}{x^2-y^2}.$$

$$4. \quad \frac{a^2+b^2}{2} - \frac{(a+b)^2}{4}. \quad \text{Ans. } \frac{2(a^2+b^2) - (a^2+2ab+b^2)}{4} = \frac{(a-b)^2}{4},$$

after removing parentheses and combining.

$$5. \quad \frac{a^2}{a^2-1} + \frac{a}{a-1} - \frac{a}{a+1}. \quad \text{Ans. } \frac{a^2+2a}{a^2-1}.$$

$$6. \quad \frac{4m^2+1}{4m^2} - \frac{3m-1}{12m^2} + \frac{1-12n}{12n}. \quad \text{Ans. } \frac{n+m^2}{12m^2n}.$$

$$7. \frac{y}{(x+y)^2} + \frac{y}{x^2-y^2} - \frac{1}{x+y}.$$

$$\text{Ans. } \frac{y(x-y) + y(x+y) - (x^2-y^2)}{(x+y)^2(x-y)} = \frac{2xy - x^2 + y^2}{x^3 + x^2y - xy^2 - y^3}.$$

$$8. \frac{x}{x+1} + \frac{x}{1-x} + \frac{3x}{x^2-1}.$$

$$\text{Ans. } \frac{x}{x^2-1}.$$

MULTIPLICATION OF FRACTIONS.

544. Multiplication, in fractions, is the process of finding a fractional part of a fraction. Thus, $\frac{3}{4} \times \frac{1}{2}$ means $\frac{1}{2}$ of $\frac{3}{4}$. One-half of $\frac{3}{4}$ inch, for example, is $\frac{3}{8}$ inch; $\frac{3}{4}$ of $\frac{3}{4}$ inch is $\frac{9}{16}$, or $\frac{6}{12}$, inch. The result in each case is the same as that which would be obtained by finding the product of the numerators and writing it over the product of the denominators.

545. Hence, to multiply fractions:

Rule.—*Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

546. Any number of fractions may be multiplied together. The operation may be very much shortened by resolving the terms of the fractions into their factors, and canceling. The product should be reduced to its simplest form.

EXAMPLE.—Find the product of $\frac{6a^2}{5}$, $\frac{2ab}{3c}$, and $\frac{2ac}{b^2}$.

SOLUTION.—The product of the numerators is $6a^2 \times 2ab \times 2ac = 24a^4bc$, and of the denominators, $5 \times 3c \times b^2 = 15b^2c$. Writing $24a^4bc$ over $15b^2c$, we have for the product, $\frac{24a^4bc}{15b^2c} = \frac{8a^4}{5b}$, when reduced to its lowest terms. The work is written as follows:

$$\frac{6a^2}{5} \times \frac{2ab}{3c} \times \frac{2ac}{b^2} = \frac{8a^4}{5b}. \quad \text{Ans.}$$

EXAMPLE.—Find the product of $\frac{x^2+2x}{(x-1)^2}$, x^2-1 , and $\frac{x^2-4x+4}{x^2-4}$.

SOLUTION.—First make $x^2 - 1$ a fraction by writing 1 for its denominator, thus, $\frac{x^2 - 1}{1}$; then, factoring both terms of each fraction, we

$$\text{have } \frac{x^2 + 2x}{(x-1)^2} \times \frac{x^2 - 1}{1} \times \frac{x^2 - 4x + 4}{x^2 - 4} = \frac{x(x+2)(x+1)(x-1)(x-2)(x-2)}{(x-1)^2(x-2)(x+2)} = \frac{x(x+1)(x-2)}{x-1}. \quad \text{Ans.}$$

EXAMPLE.—Find the product of $\frac{1}{a^2} - \frac{4c^2}{a}$, and $\frac{a^2}{1+2ac}$.

SOLUTION.—Performing the subtraction, $\frac{1}{a^2} - \frac{4c^2}{a} = \frac{1 - 4a^2c^2}{a^2}$.

$$\text{Multiplying, } \frac{1 - 4a^2c^2}{a^2} \times \frac{a^2}{1 + 2ac} = \frac{(1 - 4a^2c^2)a^2}{(1 + 2ac)a^2} = 1 - 2ac. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

547. Multiply the following:

$$1. \frac{3a^2bc}{5abc^2} \text{ by } \frac{10ab^2c}{8abc}. \quad \text{Ans. } \frac{2ab}{c}.$$

$$2. \frac{5x^2y}{7x} \text{ by } 21xy. \quad \text{Ans. } 15x^2y^2.$$

Find the product of:

$$3. \frac{3x^2y}{4xz^2}, \frac{5y^2z}{6xy}, \text{ and } \frac{-12x^2}{2xy^2}. \quad \text{Ans. } -\frac{15x}{4z}.$$

$$4. \frac{x^2 - y^2}{c^2 - d^2}, \frac{c - d}{(x + y)^2}, \text{ and } \frac{x^2 + y^2}{x - y}. \quad \text{Ans. } \frac{x^2 - xy + y^2}{c^2 + cd + d^2}.$$

$$5. \frac{4y}{x} - \frac{16}{xy}, \text{ and } \frac{1}{2y + 4}. \quad \text{Ans. } \frac{2y - 4}{xy}.$$

$$6. \frac{a + b}{2} + \frac{a - b}{4}, \text{ and } \frac{4}{9a^2 + 6ab + b^2}. \quad \text{Ans. } \frac{1}{3a + b}.$$

DIVISION OF FRACTIONS.

548. **Division**, in fractions, is the reverse of multiplication, and is the process we use when, given one of two fractions and their product, we are required to find the other. For example, we are required to divide $\frac{a}{4}$ by $\frac{1}{2}$. We wish to find such a fraction that, multiplied by $\frac{1}{2}$, will give $\frac{a}{4}$.

This fraction is $\frac{a}{2}$, for $\frac{a}{2} \times \frac{1}{2} = \frac{a}{4}$. Also $\frac{x}{5} \div \frac{x}{7} = \frac{7}{5}$, since $\frac{7}{5} \times \frac{x}{7} = \frac{x}{5}$. If, in this case, we had inverted the divisor and multiplied, we should have had $\frac{x}{5} \times \frac{7}{x} = \frac{7}{5}$.

549. Hence, to divide by a fraction:

Rule.—*Invert the divisor, and proceed as in multiplication.*

EXAMPLE.—Divide $\frac{3a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^4y^2}$.

SOLUTION.—The divisor inverted = $\frac{10x^4y^2}{9ab^3}$.

$$\text{Hence, } \frac{3a^2b}{5x^3y} \div \frac{9ab^3}{10x^4y^2} = \frac{3a^2b}{5x^3y} \times \frac{10x^4y^2}{9ab^3} = \frac{\cancel{3} \times \cancel{10}^2 \times \cancel{a}^2 \times \cancel{b}^{\cancel{1}} \times \cancel{y}^{\cancel{2}}}{\cancel{5} \times \cancel{9}^3 \times \cancel{x}^{\cancel{3}} \times \cancel{y}^{\cancel{1}}} = \frac{2xy}{3b^2}. \text{ Ans.}$$

EXAMPLE.—Divide $x^2 + 2x + 1$ by $\frac{x+1}{x-1}$.

SOLUTION.—By Art. 529, $(x^2 + 2x + 1) \div \frac{x+1}{x-1} = \frac{x^2 + 2x + 1}{1} \times$

$$\frac{x-1}{x+1} = \frac{(x+1)(x+1)(x-1)}{\cancel{x+1}} = x^2 - 1. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

550. Divide the following:

- $\frac{9x^3 - 3x^4}{24}$ by $\frac{3x}{8}$. Ans. $\frac{3x^2 - x^3}{3}$.
- $\frac{ab - bx}{a + z}$ by $\frac{ac - cx}{a + z}$. Ans. $\frac{b}{c}$.
- $\frac{1 - 8b^2 + 16b^4}{1 + 2b}$ by $\frac{1 - 4b^2}{3a}$. Ans. $3a(1 - 2b)$.
- $6a^2cd - 6abcd$ by $\frac{6acd}{a^2 + ab + b^2}$. Ans. $a^3 - b^3$.

MIXED QUANTITIES AND COMPLEX FRACTIONS.

551. An **integral expression** is one containing neither fractions nor negative exponents. The expression $a^2 + 2ab$ is integral, but the expressions $a^2 + \frac{1}{2ab}$, $2a^{-2}$, $\frac{3}{a^2 + b}$ are not. The expression $2a^{-2}$ is only another way of writing $\frac{2}{a^2}$.

The use of negative exponents will be explained in subsequent paragraphs.

552. The **integral part** of an expression is that part which, if taken by itself, would be an integral expression.

553. A **mixed quantity** is an expression containing both integral and fractional parts, as $2a^2 - \frac{c+d}{4}$. Considering the integral part, $2a^2$, as a fraction with a denominator 1 (Art. 529), a mixed quantity becomes simply the indicated addition or subtraction of two fractions; thus $2a^2 - \frac{c+d}{4} = \frac{2a^2}{1} - \frac{c+d}{4}$.

554. A fraction may be reduced to either an entire or mixed quantity by dividing the numerator by the denominator, provided the division be possible. It frequently happens that by performing the indicated division, the fraction will be reduced to a simpler form. The case of reducing a fraction to an entire quantity was taken up in Art. 535.

EXAMPLE.—Simplify $\frac{4x^2 + 12x - 1}{2x + 3}$.

SOLUTION.—Performing the indicated division,

$$\begin{array}{r} 2x+3 \overline{) 4x^2 + 12x - 1} \quad (2x+3 - \frac{10}{2x+3} \text{ Ans.} \\ \underline{4x^2 + 6x} \\ 6x - 1 \\ \underline{6x + 9} \\ -10 \end{array}$$

555. Mixed quantities are frequently more convenient to handle as fractions.

To reduce a mixed quantity to a fraction:

Rule.—Write the integral part with a denominator 1, and perform the indicated addition or subtraction.

EXAMPLE.—Reduce $x^2 + xy + y^2 - \frac{b}{x-y}$ to a fraction.

SOLUTION.— $x^2 + xy + y^2 - \frac{b}{x-y} = \frac{x^2 + xy + y^2}{1} - \frac{b}{x-y}$; subtracting the second fraction from the first gives

$$\frac{(x^2 + xy + y^2)(x-y) - b}{x-y} = \frac{x^3 - y^3 - b}{x-y} \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

556. Solve the following:

1. Reduce $\frac{a^2c + b^3}{c}$ to a mixed quantity. Ans. $a^2 + \frac{b^3}{c}$.

2. Simplify $\frac{x^3 + 4xy + 5y^2 - 3x}{x + 2y}$. Ans. $x + 2y - 3 + \frac{y^2 + 6y}{x + 2y}$.

3. Reduce $x + 3 - \frac{7x + 3}{2x + 1}$ to a fraction. Ans. $\frac{2x^2}{2x + 1}$.

4. From $3a + \frac{a + b}{d}$ subtract $a - \frac{a - b}{d}$.
Ans. $2a + \frac{2a}{d} = \frac{2a(d + 1)}{d}$.

5. Divide $m + n - \frac{2n}{m - n}$ by $m - n - \frac{2n}{m + n}$. Ans. $\frac{m + n}{m - n}$.

SUGGESTION.—First reduce the mixed quantities to fractions.

557. A **complex fraction** is one which contains frac-

tions in one or both of its terms. Thus, $\frac{a + \frac{x}{y}}{a + x}$, $\frac{a - b}{\frac{x}{y}}$, and

$\frac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

558. Complex fractions can be reduced by performing the indicated division; thus, $\frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \times \frac{4}{3} = \frac{5}{6}$. A much simpler way is to multiply both terms by the least common denominator of the fractions contained. Thus, $\frac{\frac{5}{8} \times 8}{\frac{3}{4} \times 8} = \frac{5}{6}$.

559. Hence, to simplify a complex fraction:

Rule.—Multiply both terms by the common denominator of the fractional parts.

EXAMPLE.—Simplify $\frac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{y} - \frac{1}{x}}$.

SOLUTION.—The common denominator of the fractional parts is xy . Multiplying each term by this, we have

$$\frac{\frac{x}{y} \times xy - \frac{y}{x} \times xy}{\frac{1}{y} \times xy - \frac{1}{x} \times xy} = \frac{x^2 - y^2}{x - y} = x + y. \quad \text{Ans.}$$

The multiplication can frequently be performed mentally, without writing the common denominator, at the same time canceling common factors.

EXAMPLE.—Simplify
$$\frac{1}{1 + \frac{a}{1 + a + \frac{2a^2}{1 - a}}}$$

SOLUTION.—This is the case of a complex fraction in which the denominator is itself a complex fraction.

First, consider the part
$$\frac{a}{1 + a + \frac{2a^2}{1 - a}}$$

Multiplying both terms by $1 - a$, we have

$$\frac{a(1 - a)}{(1 + a)(1 - a) + 2a^2} = \frac{a - a^2}{1 - a^2 + 2a^2} = \frac{a - a^2}{1 + a^2}$$

The fraction thus becomes
$$\frac{1}{1 + \frac{a - a^2}{1 + a^2}}$$

Multiplying both terms by $1 + a^2$, the common denominator, we have

$$\frac{1 + a^2}{1 + a^2 + a - a^2} = \frac{1 + a^2}{1 + a}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

560. Simplify the following:

1.
$$\frac{\frac{3ac^2}{16}}{\frac{24}{c}}$$
 Ans. $\frac{ac^2}{128}$

2.
$$\frac{1 + \frac{a}{c}}{c - \frac{a^2}{c}}$$
 Ans. $\frac{1}{c - a}$

3.
$$\frac{2\frac{1}{2}}{8 - 2x + \frac{x^2}{8}}$$
 Ans. $\frac{23}{(8 - x)^2}$

SUGGESTION.— $2\frac{1}{2}$ means $2 + \frac{1}{2}$. Hence, for the numerator multiply 2 by the least common denominator 8, and add 7.

4.
$$\frac{1}{x + \frac{1}{1 + \frac{x+1}{8-x}}}$$
 Ans. $\frac{4}{3x+3}$

THEORY OF EXPONENTS.

561. An exponent has already been defined in Art. **444.** In addition to positive integral exponents, as there defined, we frequently have to deal with negative and with fractional exponents. Thus, in the quantities a^3 , a^{-3} , and $a^{\frac{1}{3}}$, we have the positive integral exponent 3, the negative exponent -3 , and the fractional exponent $\frac{1}{3}$. The rules for positive and integral exponents apply also to negative and fractional exponents.

562. Since letters may represent numbers, they may be used for exponents, the same as figures. Thus, a^n means that a is to be taken as many times as a factor as there are units in n , or $a \times a \times a$, etc., to n factors. Such exponents are called **literal exponents**. Fractional, negative, and literal exponents are all read by using the word exponent.

Thus, $a^{\frac{1}{3}}$, a^{-4} , $a^{\frac{n}{m}}$, and a^n are read, " a , exponent $\frac{1}{3}$," " a , exponent minus 4," " a , exponent n over m ," and " a , exponent n ," respectively.

563. A **fractional exponent** is an expression of a root or of a power and a root combined. The numerator of the fraction denotes the power, and the denominator denotes the root. Thus, $\sqrt[n]{a}$ may be written $a^{\frac{1}{n}}$. Suppose we wish to find the cube root of a^6 ; instead of writing it $\sqrt[3]{a^6} = a^2$, we may write it $a^{\frac{6}{3}} = a^2$. Hence,

The numerator of a fractional exponent denotes a power, and the denominator, a root.

For example, $a^{\frac{1}{2}} = \sqrt{a}$; $c^{\frac{3}{4}} = \sqrt[4]{c^3}$; $x^{\frac{m}{n}} = \sqrt[n]{x^m}$; $2c^{\frac{5}{6}} = 2\sqrt[6]{c^5}$, the exponent $\frac{m}{n}$ applying only to the c .

564. The meaning of **negative exponents** may be illustrated as follows: Let it be required to divide a^4 by a^2 .

By Art. **502**, we have, $\frac{a^4}{a^2} = a^{4-2} = a^2$; likewise, $a^2 \div a^2 = \frac{a^2}{a^2} = a^{2-2} = a^0$; also, $a^0 \div a^2 = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$, etc. From this, it will be seen that $\frac{a^0}{a^2} = a^{-2}$; but, by Art. **503**, $a^0 = 1$, so

that 1 may be written for a^0 , in this expression, thus: $\frac{a^0}{a^2} = \frac{1}{a^2} = a^{-2}$. Hence,

A quantity affected with a negative exponent denotes the reciprocal of the same quantity affected with an equal positive exponent. (Art. 530.)

565. Since in $\frac{1}{a^2} = a^{-2}$, the a^{-2} changes to a^2 when placed in the denominator, we may state the following principle:

A factor may be changed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its exponent be changed.

For example, $\frac{n^{-1}}{ab} = \frac{1}{abn^1}$; $\frac{n}{ab^{-1}} = \frac{nb^1}{a}$; $\frac{x^{-2}}{5y^{-1}} = \frac{y}{5x^2}$, etc. In the last, the positive exponent 1 of the y is not written.

EXAMPLE.—Express, with positive exponents,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-3}c^{-1} + a^2b^{-2}.$$

SOLUTION.—Since these terms may be taken as fractions, with 1 for the denominators, we have, by transferring the letters with negative exponents to the denominators,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-3}c^{-1} + a^2b^{-2} = \frac{c^3}{ab^2} + \frac{1}{a^2b^3c^1} + \frac{a^2}{b^2}. \text{ Ans.}$$

566. The student must note very carefully that factors of an entire term only can be changed from numerator to denominator, or vice versa, and that when thus changed they become factors of the whole of the other term. Thus, in $\frac{a}{bc^{-2} + d}$, c^{-2} can not be transferred to the numerator by merely changing the sign of the exponent. The exponent may, however, be made positive by multiplying both terms by c^2 ; thus, $\frac{a \times c^2}{(bc^{-2} + d) \times c^2} = \frac{ac^2}{b + c^2d}$. In $\frac{ac^{-2}}{b + d}$, if we transfer the c^{-2} , it becomes $\frac{a}{c^2(b + d)}$, c^2 becoming a factor of the entire denominator.

EXAMPLE.—Clear $x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1}-x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2+b}$ of negative exponents

SOLUTION.—Treat each term of the expression separately. $x^2y^{-2}z^{-1} = \frac{x^2y^{-2}z^{-1}}{1}$; changing the factors with negative exponents to the denominator, and at the same time changing the signs of the exponents, we have $\frac{x^2}{y^2z}$. In $\frac{2xy}{y^{-1}-x^3}$, y^{-1} is not a factor of the whole denominator, so we must multiply both terms of the fraction by the reciprocal of y^{-1} or y ; thus, $\frac{2xy \times y}{(y^{-1}-x^3) \times y} = \frac{2xy^2}{1-x^3y}$. In $\frac{3a^{-1}b^{-2}c^3}{a^2+b}$, a^{-1} and b^{-2} are factors of the entire numerator, so we write them as factors of the entire denominator, with the signs of the exponents changed; thus, $\frac{3a^{-1}b^{-2}c^3}{a^2+b} = \frac{3c^3}{ab^2(a^2+b)} = \frac{3c^3}{a^3b^2+ab^3}$. Hence, $x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1}-x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2+b} = \frac{x^2}{y^2z} + \frac{2xy^2}{1-x^3y} - \frac{3c^3}{a^3b^2+ab^3}$. Ans.

EXAMPLE.—Solve the following:

$$a^3 \times a^{-1}; \quad n \times n^{-\frac{1}{2}}; \quad 2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}}; \quad c^{\frac{n}{m}} + c^{\frac{2n}{m}}; \quad x^2 + \sqrt[4]{x^2}.$$

Write the answers with positive exponents.

SOLUTION.— $a^3 \times a^{-1} = a^{3+(-1)} = a^{3-1} = a^2$. Ans.

$n \times n^{-\frac{1}{2}} = n^{1+(-\frac{1}{2})} = n^{1-\frac{1}{2}} = n^{\frac{1}{2}}$. Ans.

$$2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}} = \frac{2c^{-\frac{2}{3}}}{1} \times -\frac{1}{3\sqrt[3]{c^2}} = \frac{2}{c^{\frac{2}{3}}} \times -\frac{1}{3c^{\frac{2}{3}}} = -\frac{2 \times 1}{c^{\frac{2}{3}} \times 3c^{\frac{2}{3}}} = -\frac{2}{3c^{\frac{2}{3}+\frac{2}{3}}} = -\frac{2}{3c^{\frac{4}{3}}}$$
. Ans.

$$c^{\frac{n}{m}} + c^{\frac{2n}{m}} = c^{\frac{n}{m}} \frac{2n}{m} = c^{\frac{n-2n}{m}} = c^{\frac{-n}{m}} = c^{-\frac{n}{m}} = \frac{1}{c^{\frac{n}{m}}}$$
. Ans.

$$x^2 + \sqrt[4]{x^2} = x^2 + x^{\frac{1}{2}} = x^{2-\frac{1}{2}} = x^{\frac{4}{2}-\frac{1}{2}} = x^{\frac{3}{2}}$$
. Ans.

567. A letter may be raised to any power by multiplying its exponent by the index of the power. Thus, $(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2 \times 3} = x^6$, $(a^2b^3)^5 = a^{2 \times 5}b^{3 \times 5} = a^{10}b^{15}$, $(a^2)^{\frac{1}{2}} = a$, $(a^2)^{-\frac{1}{2}} = a^{-1}$, etc.

EXAMPLE.—Find the values of the following: $(a^{-1})^{-\frac{1}{2}}$; $(cd^{-2})^{\frac{3}{4}}$; $(x^a)^{-b} + (x^{-a})^{-b}$.

SOLUTION.—In the first, multiplying the exponents, we have $-1 \times -\frac{1}{2} = \frac{1}{2}$. Hence, $(a^{-1})^{-\frac{1}{2}} = a^{\frac{1}{2}}$, or $\sqrt{a}$. Ans. In like manner, $(c'd^{-2})^{\frac{1}{2}} = c^{\frac{1}{2}}d^{-1}$, Ans., since $1 \times \frac{1}{2} = \frac{1}{2}$, and $-2 \times \frac{1}{2} = -1$.

In the next one, $(x^a)^{-b} = x^{-ab}$ and $(x^{-a})^{-b} = x^{ab}$. Dividing, $x^{-ab} \div x^{ab} = x^{-ab-ab} = x^{-2ab}$. Ans.

568. A root of a letter affected with an exponent is extracted by dividing the exponent by the index of the root. Thus, $\sqrt[4]{y^{12}} = y^{\frac{12}{4}} = y^{3+0} = y^3$. From this principle the following rule may be deduced.

To extract any root of a monomial:

Rule.—*Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root. Make the sign of every even root of a positive quantity $\pm$, and the sign of every odd root of any quantity the same as that of the quantity.*

EXAMPLE.—Find the value of $\sqrt[4]{256a^4b^{12}c^8}$.

SOLUTION.—The 4th root of 256 is 4. The exponent of a in the root is $4 \div 4 = 1$; of b , $12 \div 4 = 3$; and of c , $8 \div 4 = 2$. As this is an even root of a positive quantity, the sign should be $\pm$. Hence, $\sqrt[4]{256a^4b^{12}c^8} = \pm 4ab^3c^2$. Ans.

EXAMPLE.—Find the value of $\sqrt[3]{\frac{27m^3x^9}{a^9b^6c^{12}}}$.

SOLUTION.— $\sqrt[3]{27m^3x^9} = 3mx^3$; $\sqrt[3]{a^9b^6c^{12}} = a^3b^2c^4$. The quantity is positive, and, as this is an odd root, its sign must be the same, or positive.

$$\text{Hence, } \sqrt[3]{\frac{27m^3x^9}{a^9b^6c^{12}}} = \frac{3mx^3}{a^3b^2c^4}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

569. Clear the following of negative exponents:

$$1. \quad x^2y^{-2}z^{-1}. \quad \text{Ans. } \frac{x^2}{y^2z^1}.$$

$$2. \quad 3a^{-1}b + \frac{2a}{b^{-3}c^{-1}} + c^{-1}. \quad \text{Ans. } \frac{3b}{a} + 2ab^3c + \frac{1}{c}.$$

$$3. \quad \frac{4a^{-2}(c+d)}{2c+d}. \quad \text{Ans. } \frac{4(c+d)}{a^2(2c+d)}.$$

Express the following without radical signs:

$$4. \quad \sqrt[4]{b^{-2}}. \quad \text{Ans. } (b^{-2})^{\frac{1}{4}} \text{ or } b^{-\frac{1}{2}}.$$

$$5. \quad 4a\sqrt{a^{-1}b^{-3}}. \quad \text{Ans. } 4aa^{-\frac{1}{2}}b^{-\frac{3}{2}} = 4a^{\frac{1}{2}}b^{-\frac{3}{2}}.$$

Find the values of the following:

6. $m^{\frac{1}{2}} \times m^{-\frac{1}{2}}$.

Ans. $m^{\frac{1}{2}}$.

7. $2ab^{\frac{1}{2}} \times a^{-\frac{1}{2}}b$.

Ans. $2a^{\frac{1}{2}}b^{\frac{3}{2}}$.

8. $c^{\frac{n}{2}} + \sqrt{c^{-n}}$.

Ans. c^n .

9. $2x^{-2} + (x^2)^{-\frac{1}{2}}$.

Ans. $2x^{-1}$.

10. $\left(c d^{-\frac{n}{m}}\right)^{2m} \times \sqrt{d^{4n}}$.

Ans. c^{2m} .

EQUATIONS.

DEFINITIONS.

570. As defined in Art. 437, an **equation** is a statement of equality between two expressions, as $x + 6 = 14$.

Every equation has two parts, called the **first** and the **second member**. The first member is the part on the left of the sign of equality, and the second member the part on the right of that sign. In $x + 6 = 14$, $x + 6$ is the first member, and 14 is the second member.

571. Equations usually consist of **known** and **unknown quantities**; that is, of quantities whose values are given, and of quantities whose values are not given, but are to be found. Thus, in $x + 6 = 14$, 6 and 14 are known quantities, and x is unknown; but since by the statement of the equation, $x + 6$ must equal 14, x must have such a value that when added to 6 the sum will be 14. Hence, the value of x is fixed for this particular case, and in a similar manner the value of a single unknown quantity in any equation is fixed by the relations that it bears to the known quantities, and this value can usually be found.

572. To **solve an equation** is to find the value of the unknown quantity. This is done by a series of **transformations** by which the first member becomes the unknown quantity, and the second member becomes a known quantity, which is, therefore, the value of the unknown quantity.

TRANSFORMATIONS.

573. In transforming an equation, the equality of its members must be preserved; otherwise the existing relations between the known and unknown quantities will be destroyed. Transformations are based upon the following principles:

574. In any equation:

I. The same quantity may be added to both members. For example, if 2 be added to both members of $x^2 = 16$, the members of the resulting equation, $x^2 + 2 = 18$, will be equal.

II. The same quantity may be subtracted from both members. Thus, if $x^2 = 16$, then $x^2 - 2 = 14$.

III. Both members may be multiplied or both divided by the same quantity. Thus, if $x^2 = 16$, then $2x^2 = 32$ and $\frac{x^2}{2} = 8$.

IV. Both members may be raised to the same power. Thus, if $x^2 = 16$, then $x^4 = 256$.

V. Like roots of both members may be extracted. Thus, if $x^2 = 16$, then $x = 4$.

A little thought will show that none of these operations will destroy the equality of the members. In the equation $16 = 16$, for example, by I, $16 + 2 = 16 + 2$; by II, $16 - 2 = 16 - 2$; by III, $16 \times 2 = 16 \times 2$, etc. It is to be observed, however, that after any transformation, the *members* do not equal their original values.

575. To **transpose** a term in an equation is to change it from one member of an equation to the other. *A term may be transposed to the other member of an equation, if its sign be changed.* Thus, in the equation $2x + 5 = 13$, let it be required to transpose $+5$ to the second member; changing its sign, we have $2x = 13 - 5$, or $2x = 8$. For, subtract 5 from both members and we have $2x + 5 - 5 = 13 - 5$, or $2x = 8$. Suppose we had $2x - 5 = 13$; changing the sign of -5 , and placing it in the second member, we have $2x = 18$; for $2x - 5 + 5 = 13 + 5$, or $2x = 18$.

576. *When the same term appears in both members of an equation, with the same sign, it may be dropped from each.* This is called **cancelation**. Thus, in $x + a = 16 + a$ we may cancel a , and have $x = 16$, for subtracting a from both members $x + a - a = 16 + a - a$; hence, $x = 16$. In $x - a = 16 - a$, $x - a + a = 16 - a + a$, and $x = 16$. But, in $x - a = 16 + a$, the a 's will not cancel, since they have different signs.

577. Changing Signs.—It is sometimes desirable to change the sign of a term in one of the members of an equation. This may be effected by multiplying both members by -1 . This gives the same result as changing the signs of all the terms of both members; thus, $x + a + 3 = a - x - 7$ may be changed to $-x - a - 3 = -a + x + 7$. According to Art. 574, III, this transformation does not destroy the equality of the members.

578. Clearing of fractions is usually necessary before performing any operations.

Rule.—*To clear an equation of fractions, multiply each term of the equation by the common denominator of all the fractions.*

EXAMPLE.—Clear of fractions $x + \frac{x}{2} + \frac{3x}{4} - \frac{2x}{6} = 10$.

SOLUTION.—The common denominator of all the fractions is 12; multiplying each term by 12, we have

$$12x + 6x + 9x - 4x = 120.$$

EXAMPLE.—Clear of fractions $\frac{2x}{x+2} = \frac{1}{2} - \frac{3x+2}{x^2-4}$.

SOLUTION.—The common denominator is $2(x^2-4)$; multiplying through by this we get $4x(x-2) = (x^2-4) - 2(3x+2)$; removing the parentheses this becomes $4x^2 - 8x = x^2 - 4 - 6x - 4$.

Where a fraction is preceded by a minus sign, care must be taken to change the sign of every term of the numerator when clearing of fractions. See Art. 542.

EXAMPLES FOR PRACTICE.

579. Clear the following equations of fractions:

$$1. \quad x + \frac{3x}{4} + \frac{5}{7} = 16 - \frac{2}{x}. \quad \text{Ans. } 28x^2 + 21x^2 + 20x = 448x - 56.$$

$$2. \quad \frac{x}{4} - \frac{x-3}{2} = \frac{a}{6}. \quad \text{Ans. } 3x - 6x + 18 = 2a.$$

$$3. \quad \frac{x}{a-b} - x = \frac{a-b}{a+b} - 1. \\ \text{Ans. } ax + bx - a^2x + b^2x = a^2 - 2ab + b^2 - a^2 + b^2.$$

$$4. \quad \frac{1}{a-b} = \frac{x}{a-b} - \frac{a+b}{x}. \quad \text{Ans. } x = x^2 - a^2 + b^2.$$

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

580. **Simple equations**, when reduced to their simplest form, contain the first power only, of the unknown quantity. When there is but one unknown quantity, it is usually represented by x . Numbers and the *first* letters of the alphabet are used for known quantities.

581. To solve simple equations with one unknown quantity:

Rule.—*Clear of fractions. Remove all signs of aggregation. Transpose all terms containing the unknown quantity to the first member; all others to the second member. Combine the first member into one term, and simplify the second member. Divide both members by the coefficient of the unknown quantity.*

In some cases, this order of operations may be changed to advantage.

582. To verify the result, substitute the value of the unknown quantity in the original equation, which should then reduce so that both members will be alike. When this occurs the equation is said to be **satisfied**.

EXAMPLE.—Solve $2x + 5 = 25 - 3x$.

SOLUTION.—Transposing the unknown quantities to the first member, we have $2x + 3x + 5 = 25$.

Transposing the known quantities to the second member, we have

$$2x + 3x = 25 - 5.$$

Combining like terms, $5x = 20.$

Dividing by 5, $x = 4.$ Ans.

Now, if the substituting of 4 for x will satisfy the original equation, we know our answer is correct. Thus, substituting 4 for x , we have

$$2 \times 4 + 5 = 25 - 3 \times 4.$$

$$8 + 5 = 25 - 12.$$

$$13 = 13.$$

Hence the equation is satisfied, and our result is correct.

EXAMPLE.—Solve $16 - x - \{7x - [8x - (9x - \overline{3x - 6x})]\} = 0.$

SOLUTION.—Removing the symbols of aggregation (Art. 484),

$$16 - x - 7x + 8x - 9x + 3x - 6x = 0.$$

Transposing 16 to the second member,

$$-x - 7x + 8x - 9x + 3x - 6x = -16.$$

Combining like terms, $-12x = -16.$

Dividing both members by -12 ,

$$x = \frac{4}{3} = 1\frac{1}{3}. \text{ Ans.}$$

EXAMPLE.—Solve $\frac{2x+2}{2} + \frac{1}{4} = \frac{8-6x}{5} + \frac{2(6x+7)}{8}.$

SOLUTION.—Simplifying the first and the last term of the equation, it becomes,

$$x + 1 + \frac{1}{4} = \frac{8-6x}{5} + \frac{6x+7}{4}.$$

Clearing of fractions we have,

$$20x + 20 + 5 = 32 - 24x + 30x + 35.$$

Transposing, $20x + 24x - 30x = 32 + 35 - 20 - 5.$

Combining, $14x = 42.$

Dividing by 14, $x = 3.$

EXAMPLE.—Solve $\frac{3}{1-x} - \frac{2}{1+x} + \frac{1}{1-x^2} = 0.$

SOLUTION.—Clearing of fractions by multiplying by $1 - x^2$,

$$3(1+x) - 2(1-x) + 1 = 0.$$

$$3 + 3x - 2 + 2x + 1 = 0.$$

Uniting terms, $5x = -2.$

$$x = -.4. \text{ Ans.}$$

NOTE.— 0 multiplied or divided by any number = 0.

583. Literal equations are those in which the known quantities are represented partially or wholly by letters. In solving such equations, we can not always combine the unknown quantities into one term.

EXAMPLE.—Solve $2ax - 3b = x + c - 3ax$.

SOLUTION.—Transposing the terms containing the unknown quantities to the first member and the remaining terms to the second member, and combining like terms,

$$5ax - x = 3b + c.$$

Factoring $5ax - x$ to bring x alone in the first member,

$$(5a - 1)x = 3b + c.$$

The coefficient of x is now $5a - 1$, this being considered as one quantity.

Dividing by $5a - 1$, $x = \frac{3b + c}{5a - 1}$. Ans.

PROOF.—Since the original equation is equivalent to $5ax - x = 3b + c$, it will be sufficient to satisfy this equation. Hence, substituting the value of x ,

$$\frac{5a(3b + c)}{5a - 1} - \frac{3b + c}{5a - 1} = 3b + c, \text{ or } \frac{(5a - 1)(3b + c)}{5a - 1} = 3b + c.$$

Canceling the $5a - 1$, $3b + c = 3b + c$.

EXAMPLE.—Solve $\frac{3x + 1}{x + 1} = \frac{3bx - 2a + c}{b(x + 1) - a}$.

SOLUTION.—Clearing of fractions,

$$(3x + 1)[b(x + 1) - a] = (x + 1)(3bx - 2a + c), \text{ or}$$

$$3bx(x + 1) - 3ax + b(x + 1) - a = 3bx(x + 1) - (2a - c)(x + 1).$$

Canceling $3bx(x + 1)$ from both members,

$$-3ax + bx + b - a = -2ax + cx - 2a + c.$$

Transposing and uniting terms,

$$-ax + bx - cx = -a - b + c.$$

Changing signs and factoring,

$$(a - b + c)x = a + b - c.$$

$$x = \frac{a + b - c}{a - b + c}.$$

EXAMPLES FOR PRACTICE.

584. Solve the following:

1. $16 - 3x = 13 - 6x$.

Ans. $x = -1$.

2. $3(4x - 5) + 6 = 1 + 2x$.

Ans. $x = 1$.

3. $6(5 - 2x) = 6 - 2(x - 2)$.

Ans. $x = 2$.

$$4. \quad \frac{2x}{3} - \frac{4x}{3} = 5 - \frac{3x}{4}.$$

$$\text{Ans. } x = 60.$$

$$5. \quad \frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4}.$$

$$\text{Ans. } x = 41.$$

$$6. \quad \frac{x}{3} - \frac{x^2 - 5x}{3x - 7} = \frac{2}{3}.$$

$$\text{Ans. } x = -7.$$

$$7. \quad \frac{5-2x}{x+1} - \frac{3-2x}{x+4} = 0.$$

$$\text{Ans. } x = 4\frac{1}{2}.$$

$$8. \quad 2x - 4a = 3ax + a^2 - a^2x.$$

$$\text{Ans. } x = \frac{a^2 + 4a}{a^2 - 3a + 2}.$$

$$9. \quad \frac{ax + 2x}{5a} - \frac{a^2 + 4a + 4}{4b} = 0.$$

$$\text{Ans. } x = \frac{5a^2 + 10a}{4b}.$$

SUGGESTION.—Transposing the second term to the second member,

$$\frac{ax + 2x}{5a} = \frac{a^2 + 4a + 4}{4b} = \frac{(a+2)^2}{4b}.$$

Multiplying both sides by $5a$,

$$ax + 2x = \frac{5a(a+2)^2}{4b}.$$

Solving for x ,

$$x = \frac{5a(a+2)^2}{(a+2)4b} = \frac{5a(a+2)}{4b} = \frac{5a^2 + 10a}{4b}.$$

$$10. \quad \frac{a(c^2 + x^2)}{cx} = ab + \frac{ax}{c}.$$

$$\text{Ans. } x = \frac{c}{b}.$$

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

585. There are two steps in the solution of problems by algebra:

First.—The relations which exist between the known and the unknown quantities, that is, between those whose values are given in the problem and those whose values are required, must be stated by one or more equations. This is called the **statement** of the problem.

Second.—The resulting equation or equations must be solved, giving the values of the required quantities.

It will thus be seen that by the algebraic method, the answer to a problem is used in the solution and operated upon as though it were a known quantity, which is one great advantage over the arithmetical method.

586. The ability to state a problem by means of an equation depends upon the ingenuity of the operator and his ability to reason, rather than upon his knowledge of algebra. No definite rule can be given for making the statement, but in general, where there is only one unknown quantity in a problem:

Decide what quantity it is whose value is to be found. This will be the unknown quantity, or the answer. Then represent the unknown quantity by x and form an equation that will indicate the relations between the known and the unknown quantities as stated in the problem.

The equation will also indicate the operations that would be performed in proving the statement made in the problem were the answer known. Hence, the equation may often be formed by noticing what operations would be performed upon the answer in proving.

587. The following examples are illustrations of the statement and the solution of algebraic problems; they should be studied carefully.

EXAMPLE.—Find such a number that, when 14 is added to its double, the sum shall be 30.

SOLUTION.—The quantity whose value is required is the number itself. As this is the unknown quantity, let x = the number, whence $2x$ must be double the number. Now the problem states that when 14 is added to double the number the sum will be 30. In other words, when 14 is added to $2x$, the sum will be 30. Hence, the statement of the problem in the form of an equation is,

$$2x + 14 = 30;$$

whence, solving,

$$x = 8. \quad \text{Ans.}$$

EXAMPLE.—Find a number which, when multiplied by 4, will exceed 40 as much as it is now below 40.

SOLUTION.—Let x = the required number, which, when multiplied by 4, becomes $4x$. According to the conditions of the problem, the amount by which 4 times the required number, or $4x$, exceeds 40 is equal to the amount that the number itself, or x , is below 40.

But $4x - 40$ is the amount by which $4x$ exceeds 40, and $40 - x$ is the amount by which x is below 40.

Hence, by the conditions, we have the statement,

$$4x - 40 = 40 - x.$$

Transposing and uniting,

$$5x = 80,$$

or

$$x = 16. \quad \text{Ans.}$$

EXAMPLE.—Two loads of brick together weigh 4,000 lb.; but if 500 lb. be transferred from the smaller to the larger load, the latter will weigh 7 times as much as the former. How much does each load weigh?

SOLUTION.—If the weights of the two loads were known and it was desired to prove the correctness of the example, we should add 500 lb. to the weight of the larger load and subtract 500 lb. from the weight of the smaller load, as stated in the example. The larger load should then weigh 7 times as much as the smaller. To obtain the equation the same thing is done by letting x = the weight of one load, whence $4,000 - x$ is equal to the weight of the other load.

Let x = the weight of the smaller load.

Then, $4,000 - x$ = the weight of the larger load.

Also, $x - 500$ = the weight of the smaller load after transferring 500 lb.

And $4,000 - x + 500$ = the weight of the larger load after transferring 500 lb.

By the conditions, the larger load now weighs 7 times as much as the smaller.

Hence, $7(x - 500) = 4,000 - x + 500$.

Solving, $7x - 3,500 = 4,500 - x$,

or $8x = 8,000$;

whence, $x = 1,000$ lb. = weight of smaller load, } Ans.
and $4,000 - x = 3,000$ lb. = weight of larger load. }

PROOF.— $1,000 - 500 = 500$ = weight of the smaller load, and $3,000 + 500 = 3,500$ = weight of the larger load after the 500 pounds have been transferred; $3,500 \div 500 = 7$.

Until the student has obtained considerable proficiency in solving problems of this kind, it is a good plan to prove all problems.

EXAMPLE.—The circumference of the fore wheel of a carriage is 10 feet, and of the rear wheel, 12 feet. What distance has the carriage traveled, when the fore wheel has made 8 more turns than the hind wheel?

SOLUTION.—In this example the distance traveled is not known, but is required to be found. Suppose that the distance is known, and that it is equal to x feet, and that we wish to see whether the statement is true that the fore wheel makes 8 more revolutions than the rear wheel in passing over x feet. The number of revolutions of the fore wheel is evidently $\frac{x}{10}$, and of the rear wheel, $\frac{x}{12}$. The example states that the difference between them is equal to 8.

Hence, $\frac{x}{10} - \frac{x}{12} = 8. \quad (1)$

Solving for x , $12x - 10x = 960$,
 or $2x = 960$
 and $x = 480$ feet. Ans.

PROOF.— $\frac{480}{10} = 48 = \text{revolutions of fore wheel.}$

$\frac{480}{12} = 40 = \text{revolutions of hind wheel.}$

$48 - 40 = 8.$ Compare this proof with (1).

EXAMPLE.—A water cistern connected with three pipes can be filled by one of them in 80 minutes, by another in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

SOLUTION.—Here the unknown quantity is the number of minutes required to fill the cistern by all three pipes together. Supposing this to be x minutes, the example may be proved by noticing that the sum of the quantities of water flowing through each pipe separately in a given length of time, as 1 minute, must be equal to the quantity flowing through all three together in the same length of time. According to the problem, the quantity discharged by the first pipe in 1 minute would be $\frac{1}{80}$, by the second $\frac{1}{200}$, and by the third $\frac{1}{300}$ of the contents of the cistern. In like manner the quantity discharged by all three at once in 1 minute would be $\frac{1}{x}$. Then, if the example is stated correctly, we must have

$$\frac{1}{80} + \frac{1}{200} + \frac{1}{300} = \frac{1}{x}.$$

Clearing of fractions,

$$\begin{aligned} x(30 + 12 + 8) &= 2,400, \\ \text{or } 50x &= 2,400; \\ \text{whence, } x &= 48 \text{ minutes. Ans.} \end{aligned}$$

EXAMPLE.—A man rows a boat a certain distance *with* the tide, at the rate of $6\frac{1}{2}$ miles an hour, and returns at the rate of $3\frac{1}{2}$ miles an hour, *against* a tide half as strong. If the man is pulling at a uniform rate, what is the velocity of the stronger tide?

SOLUTION.—If the following statement is not clear, the student should reason it out for himself in a manner similar to that used in the last three examples.

Let x = number of miles per hour that the stronger tide is running;
 then $\frac{x}{2}$ = number of miles per hour that the weaker tide is running.

Hence, $6\frac{1}{2} - x$ and $3\frac{1}{2} + \frac{x}{2}$ are expressions for the rate at which the

man is pulling. But, as he is pulling at a constant rate all the time, these expressions must be equal. Hence,

$$6\frac{1}{2} - x = 3\frac{1}{2} + \frac{x}{2},$$

$$\text{or } \frac{20}{3} - x = \frac{10}{3} + \frac{x}{2}.$$

Clearing of fractions, $40 - 6x = 20 + 3x$,

$$\text{or } -9x = -20;$$

whence, $x = 2\frac{1}{3}$ miles per hour. Ans.

EXAMPLES FOR PRACTICE.

588. Solve the following examples:

1. The greater of two numbers is four times the lesser number, and their sum is 400; what are the numbers? Ans. 80 and 320.

2. A farmer has 108 animals, consisting of horses, sheep, and cows. He has four times as many cows as horses, lacking 8, and five times as many sheep as horses, lacking 4; how many has he of each kind?

Ans. $\begin{cases} 12 \text{ horses.} \\ 40 \text{ cows.} \\ 56 \text{ sheep.} \end{cases}$

3. A can do a piece of work in 8 days, and B can do it in 10 days; in what time can they do it working together? Ans. $4\frac{1}{3}$ days.

4. Find five consecutive numbers whose sum is 150.

Ans. 28, 29, 30, 31, and 32.

5. A boat whose rate of sailing is 6 miles per hour in still water moves down a stream which flows at the rate of 3 miles per hour, and returns, making the round trip in 8 hours; how far did it go down the stream? Ans. 18 miles.

THE TRIGONOMETRIC FUNCTIONS.

DEFINITIONS.

589. Plane trigonometry treats of the solution of plane triangles.

Every triangle has six parts—three angles and three sides. When three of these parts are given, if one part at least is a side, the other three can be found. This process of finding the unknown parts of a triangle is the **solution** of the triangle.

590. The **complement** of an angle is the difference between 90° and the angle. Thus, the complement of an angle of 35° is an angle of 55° , because $90^\circ - 35^\circ = 55^\circ$. In a right-angled triangle, the right angle is 90° ; since the sum of the three angles of the triangle is 180° , the sum of the two acute angles is $180^\circ - 90^\circ = 90^\circ$. Therefore, each acute angle of a right-angled triangle is the complement of the other acute angle.

591. The **supplement** of an angle is the difference between 180° and the angle. Thus, the supplement of an angle of 35° is an angle of 145° , because $180^\circ - 35^\circ = 145^\circ$.

592. The solution of a triangle is accomplished by means of the **trigonometric functions**. These functions are the ratios of the sides of a right-angled triangle; the most important of these functions are the *sine*, *cosine*, *tangent*, and *cotangent*. These are abbreviated to *sin*, *cos*, *tan*, and *cot*.

593. In the right-angled triangle ABC , Fig. 63, the sides a , b , and c are opposite, respectively, to the angles A , B , and C . The hypotenuse is c , and C is the right angle. The short side b is adjacent to angle A and opposite angle B ; the short side a is opposite to angle A and adjacent to angle B .

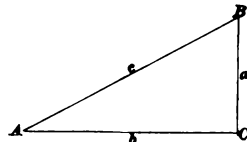


FIG. 63.

Then the trigonometric functions are defined as follows:

594. The **sine** of an angle is the quotient of the opposite side divided by the hypotenuse. Thus, in Fig. 63 $\sin A = \frac{a}{c}$.

595. The **cosine** of an angle is the quotient of the adjacent side divided by the hypotenuse. Thus, $\cos A = \frac{b}{c}$.

596. The **tangent** of an angle is the quotient of the opposite side divided by the adjacent side. Thus, $\tan A = \frac{a}{b}$.

597. The **cotangent** of an angle is the quotient of the adjacent side divided by the opposite side. Thus,

$$\cot A = \frac{b}{a}.$$

598. From the definitions of the functions, we have:

$$\sin B = \frac{b}{c}; \cos B = \frac{a}{c}; \tan B = \frac{b}{a}; \text{ and } \cot B = \frac{a}{b}.$$

Comparing the functions of angle B with those of angle A , $\sin$

$B = \cos A$ (since each is equal to $\frac{b}{c}$), $\cos B = \sin A$, $\tan$

$B = \cot A$, and $\cot B = \tan A$. It has been shown (Art.

590) that angle A is the complement of angle B . Therefore, the sine of an angle is equal to the cosine of its complement, and the tangent of an angle is equal to the cotangent of its complement. For example, $\sin 36^\circ = \cos (90^\circ - 36^\circ) = \cos 54^\circ$; $\tan 28^\circ = \cot 62^\circ$, etc.

TRIGONOMETRIC TABLES.

599. Every angular function has a different value for each of the angles between 0° and 90° . The numerical values of these functions are called natural sines, cosines, etc., and are given in the tables of Natural Sines, Cosines, Tangents, and Cotangents. In many tables, both natural and logarithmic functions are given. The student should not attempt to use the latter until he thoroughly understands logarithms. The table of natural functions, and its use, will now be explained.

600. Given an Angle, to Find Its Functions.

EXAMPLE.—Let it be required to find the sine, cosine, and tangent of an angle of $37^\circ 24'$.

SOLUTION.—Look in the table of **natural sines** along the tops of the pages and find 37° . The left-hand column is marked ($'$), meaning that the minutes are to be sought in that column, and begin with 0, 1, 2, 3, etc., up to 60. Glancing *down* this column until 24 is found, find opposite this 24 in the column marked *sine*, and headed 37° , the number .60738; then $.60738 = \sin 37^\circ 24'$. In exactly the same manner, find in

the column marked *cosine* and headed 37° , the number .79441, which corresponds to $\cos 37^\circ 24'$; or $\cos 37^\circ 24' = .79441$. So, also, find in the column marked *tangent* and headed 37° , and opposite $24'$, the number .76456; hence, $\tan 37^\circ 24' = .76456$.

601. In most of the tables published, the angles run only from 0° to 45° at the top of the page; to find an angle greater than 45° , *look at the bottom of the page and glance upwards, using the extreme right-hand column to find minutes*, which begin with 0 at the bottom and run upwards, 1, 2, 3, etc., up to 60.

EXAMPLE.—Find the sine, cosine, and tangent of $77^\circ 43'$.

SOLUTION.—Since this angle is greater than 45° , look in the tables along the bottom of the page, until the column marked 77° is found. Glancing up the column of minutes on the *right*, until $43'$ is found, find opposite $43'$ in the column marked *sine* (and 77°) at the bottom, the number .97711; this is the sine of $77^\circ 43'$, or $\sin 77^\circ 43' = .97711$. Similarly, in the column marked *cosine*, find opposite $43'$ in the right-hand column, the number .21275; this is the cosine of $77^\circ 43'$, or $\cos 77^\circ 43' = .21275$. So, also, find that 4.59283 is the tangent of $77^\circ 43'$, or $\tan 77^\circ 43' = 4.59283$.

602. Let it be required to find the sine of $14^\circ 22' 26''$.

EXPLANATION.—The sine of $14^\circ 22' 26''$ lies between $\sin 14^\circ 22'$ and $\sin 14^\circ 23'$. $\sin 14^\circ 22' = .24813$; $\sin 14^\circ 23' = .24841$; difference = .00028. Neglect for the moment the fact that the functions are decimal fractions; then the difference between $\sin 14^\circ 22'$ and $\sin 14^\circ 23'$, that is, the difference between 24841 and 24813, is 28, or, corresponding to a difference of 1' in the angle, there is a difference of 28 in the sine. Now, since $1' = 60''$, the difference between $\sin 14^\circ 22'$ and $\sin 14^\circ 22' 26''$, that is, the difference corresponding to a difference of $26''$ in the angle, must be $\frac{26}{60} \times 28 = 12.1$. Since .1 is less than .5, omit it, and we have 12 as the difference to be added to 24813. $24813 + 12 = 24825$; taking account now of the fact that the function is a decimal fraction, $\sin 14^\circ 22' 26'' = .24825$.

In all work with the tables it will always be found most convenient to neglect for the moment the decimal point,

consider the difference a whole number, and afterwards locate the decimal point in its proper position.

603. Reference to the table of functions shows that, as the angles increase in magnitude, the sines and tangents increase, while the *cosines and cotangents decrease*. In the above example, therefore, had it been required to find the *cosine* of $14^{\circ} 22' 26''$, the correction for the $26''$ would have been *subtracted* from the cosine of $14^{\circ} 22'$, instead of being added to it.

EXAMPLE.—Find the sine, cosine, and cotangent of $56^{\circ} 43' 17''$.

SOLUTION.— $\sin 56^{\circ} 43' = .83597$. $\sin 56^{\circ} 44' = .83613$. Since $56^{\circ} 43' 17''$ is greater than $56^{\circ} 43'$ and less than $56^{\circ} 44'$, the value of the sine of the angle lies between .83597 and .83613; neglecting the decimal character of the functions the difference $= 83613 - 83597 = 16$; multiplying this by the fraction $\frac{17}{60}$, $16 \times \frac{17}{60} = 4.53$. Since .53 exceeds .5, we take 5 as the difference to be *added* to 83597. Adding, $83597 + 5 = 83602$; hence, $\sin 56^{\circ} 43' 17'' = .83602$. Ans.

$\cos 56^{\circ} 43' = .54878$; $\cos 56^{\circ} 44' = .54854$; expressed as a whole number the difference $= 54878 - 54854 = 24$, and $24 \times \frac{17}{60} = 7$, nearly. Now, since the *cosine* is desired, we must *subtract* this correction from 54878; subtracting, $54878 - 7 = 54871$. Hence, $\cos 56^{\circ} 43' 17'' = .54871$. Ans.

$\cot 56^{\circ} 43' = .65646$; $\cot 56^{\circ} 44' = .65604$; difference $= 65646 - 65604 = 42$, and $42 \times \frac{17}{60} = 12$, nearly. Now, since the cotangent decreases as the angle increases, we must *subtract* this correction from 65646; thus, $65646 - 12 = 65634$. Hence, $\cot 56^{\circ} 43' 17'' = .65634$. Ans.

604. Given the Function, to Find the Corresponding Angle.—This is the reverse of the process for finding the function of the angle. If the angle corresponding to the given function is an exact number of degrees and minutes, the function will be found in the table. In such a case, we have simply to find the given function, and take the degrees from the end of the column, and the minutes at the end of the row. If the name of the function is at the top of the column, the number of degrees will be found there also, and the minutes at the left. If the name is at the bottom, the degrees will be at the bottom, and the minutes at the right.

CASE I.

605. *The function is found exactly in the table.*

EXAMPLE.—Find the angle whose sine is .24982.

SOLUTION.—In table of sines, we find .24982 in the column under 14° . Since the name of the function is at the top of the column, we take the number of degrees from the top, and the minutes from the left; thus, 14° at the top, and 28' at the left. Hence, the angle whose sine is .24982 is $14^\circ 28'$. Ans.

EXAMPLE.—Find the angle whose cotangent is .68557.

SOLUTION.—In the table of tangents, we find .68557 under 34° . The name of the function is, however, at the bottom of the page, and we must take the degrees, 55° , from that end of the column. The minutes, 34', are found in the right-hand column. Hence, the angle whose cotangent is .68557 is $55^\circ 34'$. Ans.

CASE II.

606. *The function is not found exactly in the table.*

Let it be required to find the angle whose sine is .42531.

EXPLANATION.—Referring to the table of sines, this number is found to lie between .42525, the sine of $25^\circ 10'$, and .42552, the sine of $25^\circ 11'$. Neglecting decimals, the difference between these two sines = $42552 - 42525 = 27$; the difference between 42525, corresponding to the sine of $25^\circ 10'$, and 42531, corresponding to the sine of the given angle = $42531 - 42525 = 6$. Since 27 is the difference for 1', a difference of 6 corresponds to $\frac{6}{27}$ of 1'; hence, the angle whose sine = .42531 = $25^\circ 10\frac{6}{27}'$.

Since $1' = 60''$, $\frac{6}{27}$ of a minute = $\frac{6}{27} \times 60 = 13.3''$. Therefore, the angle whose sine is .42531 = $25^\circ 10' 13.3''$.

The given function should always be compared with the function of the *angle next lower*, and the correction in seconds should be added to that angle. In the case of the sine and tangent, we take the difference between the given function and the next smaller function appearing in the tables; but with the cosine and cotangent, we take the difference between the given function and the next larger function which appears in the tables.

EXAMPLE.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of cosines, the given function is found to belong to an angle greater than 45° and, hence, must be sought for in the columns marked *cosine* at the bottom of the page. It is found between the numbers $.27060 = \cos 74^\circ 18'$ and $.27032 = \cos 74^\circ 19'$. The difference between the two is $.27060 - .27032 = .00028$, or 28, neglecting decimals. The cosine of the *smaller angle*, or $74^\circ 18'$, is $.27060$, and the difference between this and the given cosine is $.27060 - .27052 = .00008$, or 8, neglecting decimals.

Hence, $\frac{8}{28} \times 60 = 17.14'$, nearly, and the angle whose cosine is $.27052 = 74^\circ 18' 17.14''$, or $\cos 74^\circ 18' 17.14'' = .27052$. Ans.

EXAMPLE.—Find the angle whose tangent is 2.15841.

SOLUTION.—2.15841 falls between $2.15760 = \tan 65^\circ 8'$, and $2.15925 = \tan 65^\circ 9'$. The difference $= 2.15925 - 2.15760 = .00165$, or 165, considered as a whole number. $2.15841 - 2.15760 = 81$, neglecting the decimal. $\frac{81}{165} \times 60 = 29.5''$, nearly, and the angle whose tangent is 2.15841 $= 65^\circ 8' 29.5''$, or $\tan 65^\circ 8' 29.5'' = 2.15841$.

EXAMPLES FOR PRACTICE.

607. Solve the following examples:

1. Find the (a) sine, (b) cosine, and (c) tangent of $48^\circ 17'$.

$$\text{Ans. } \begin{cases} (a) .74644. \\ (b) .66545. \\ (c) 1.12172. \end{cases}$$

2. Find the (a) sine, (b) cosine, and (c) cotangent of $13^\circ 11' 6''$.

$$\text{Ans. } \begin{cases} (a) .22810. \\ (b) .97364. \\ (c) 4.26855. \end{cases}$$

3. Find the (a) sine, (b) cosine, and (c) tangent of $72^\circ 0' 1.8''$.

$$\text{Ans. } \begin{cases} (a) .95106. \\ (b) .30901. \\ (c) 3.07777. \end{cases}$$

4. (a) Of what angle is .26489 the sine, (b) of what is it the cosine, and (c) of what is it the cotangent?

$$\text{Ans. } \begin{cases} (a) 15^\circ 21' 37.2''. \\ (b) 74^\circ 38' 22.8''. \\ (c) 75^\circ 9' 49''. \end{cases}$$

5. (a) Of what angle is .68800 the sine, (b) of what the cosine, and (c) of what the tangent?

$$\text{Ans. } \begin{cases} (a) 43^\circ 28' 20''. \\ (b) 46^\circ 31' 40''. \\ (c) 34^\circ 31' 40.5''. \end{cases}$$

SOLUTION OF TRIANGLES.

RIGHT-ANGLED TRIANGLES.

608. When any side and one of the acute angles of a right-angled triangle (also called *right triangle*) are given, the remaining sides and angles may be found; also, when any two sides are given, the remaining parts may be found.

609. The following relations between the sides and angles of a right triangle are derived directly from the definitions of the trigonometric functions :

- I. *Side opposite an angle* = *hypotenuse* $\times$ *sine of angle*.
- II. *Side adjacent* = *hypotenuse* $\times$ *cosine*.
- III. *Side opposite* = *side adjacent* $\times$ *tangent*.
- IV. *Side adjacent* = *side opposite* $\times$ *cotangent*.
- V. *Hypotenuse* = $\frac{\text{side opposite}}{\text{sine}}$.
- VI. *Hypotenuse* = $\frac{\text{side adjacent}}{\text{cosine}}$.

These relations are sufficient to find the sides. The angles may be found from the sides by the relations given in Arts. 594 to 597. To show the application of these relations to the solution of triangles, a number of examples are given.

610. Case I.—*The hypotenuse and an acute angle being given, to find the remaining parts :*

EXAMPLE.—In Fig. 64, the hypotenuse AB of the right-angled triangle ACB is 24 feet and the angle A is $29^{\circ} 31'$; to find the sides AC and BC and the angle B .

NOTE.—When working examples of this kind, construct the figure, and mark the known parts. This is a great help in solving the example. Hence, in the figure draw the angle A to represent an angle of $29^{\circ} 31'$, and complete the right-angled triangle ACB , right-angled at C , as shown. Mark the angle A and the hypotenuse, as is done in the figure.

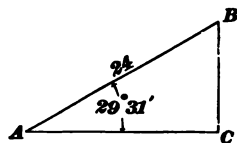


FIG. 64.

SOLUTION.—Angle $B = 90^\circ - 29^\circ 31' = 60^\circ 29'$. AC is the side adjacent to angle A . Hence, from relation II, Art. 609, AC , or side adjacent = hypotenuse $\times$ cosine = $24 \times \cos 29^\circ 31'$. In the table of Natural Cosines we find the cosine of $29^\circ 31'$ to be .87021. Therefore, $AC = 24 \times .87021 = 20.89$ feet, nearly.

To find the side BC , we use relation I, Art. 609. BC , or side opposite = hypotenuse $\times$ sine = $24 \times \sin 29^\circ 31'$. The sine of $29^\circ 31'$ is .49268; hence, $BC = 24 \times .49268 = 11.82$ feet, nearly.

$$\text{Ans. } \begin{cases} \text{Angle } B = 60^\circ 29'. \\ \text{Side } AC = 20.89 \text{ ft.} \\ \text{Side } BC = 11.82 \text{ ft.} \end{cases}$$

611. Case II.—*An acute angle and one of the short sides being given, to determine the remaining parts:*

EXAMPLE.—In Fig. 64, suppose the side AC to be 75 feet and the angle A to be $32^\circ 24'$. The sides BC , AB , and the angle B are required.

SOLUTION.—Angle $B = 90^\circ - 32^\circ 24' = 57^\circ 36'$. To find BC , we have relation III, Art. 609, BC , side opposite = side adjacent $\times$ tangent = $75 \times \tan 32^\circ 24'$. Referring to the table of Natural Tangents, the tangent of $32^\circ 24' = .63462$. $BC = 75 \times .63462 = 47.6$ ft., nearly. To find the hypotenuse AB , we use relation VI, Art. 609, hypotenuse = $\frac{\text{side adjacent}}{\text{cosine}}$. $AB = \frac{AC}{\cos A} = \frac{75}{\cos 32^\circ 24'} = \frac{75}{.84433} = 88.81$ ft.

$$\text{Ans. } \begin{cases} \text{Angle } B = 57^\circ 36'. \\ \text{Side } BC = 47.6 \text{ ft.} \\ \text{Side } AB = 88.81 \text{ ft.} \end{cases}$$

612. Case III.—*Two sides being given, to find the third side and the acute angles:*

EXAMPLE.—In the right-angled triangle ABC , Fig. 65, right-angled at C , $AC = 18$ and $BC = 15$; to find AB and the angles A and B .

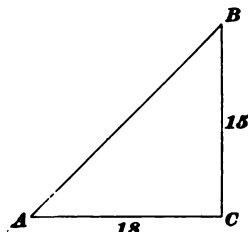


FIG. 65.

SOLUTION.—According to the definition of the tangent, Art. 596,

$$\text{tangent } A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{15}{18} = .83333.$$

Looking in the table of Tangents, the angle whose tangent is nearest to .83333 is $39^\circ 48'$. Hence, angle $A = 39^\circ 48'$, nearly. Angle $B = 90^\circ - 39^\circ 48' = 50^\circ 12'$.

To find the hypotenuse, we use relation V, Art. 609,

$$AB, \text{ hypotenuse} = \frac{\text{side opposite}}{\text{sine}} = \frac{BC}{\sin A}.$$

The sine of $39^\circ 48'$ is .64011. Hence, $AB = \frac{15}{.64011} = 23.43$.

$$\text{Ans. } \begin{cases} \text{Angle } A = 39^\circ 48'. \\ \text{Angle } B = 50^\circ 12'. \\ \text{Side } AB = 23.43. \end{cases}$$

EXAMPLE.—In the right-angled triangle ABC , Fig. 66, right-angled at C , $AC = .024967$ mile and $AB = .04792$ mile; to find the other parts.

SOLUTION.—According to the definition of the cosine, Art. 595,

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{.024967}{.04792} = .52101.$$

Referring to the table, the angle whose cosine is .52101 is $58^\circ 36'$. Therefore, angle $A = 58^\circ 36'$. Angle $B = 90^\circ - 58^\circ 36' = 31^\circ 24'$.

To find side BC , relation III, Art. 609, is used.

Side opposite $A = \text{side adjacent} \times \tan A$, or $BC = AC \times \tan 58^\circ 36' = .024967 \times 1.63826 = .0409$ mile.

$$\text{Ans. } \begin{cases} \text{Angle } A = 58^\circ 36'. \\ \text{Angle } B = 31^\circ 24'. \\ BC = .0409 \text{ mile.} \end{cases}$$

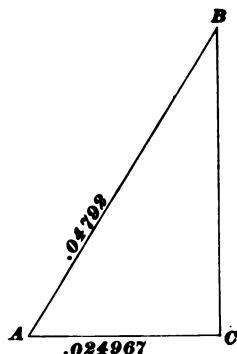


FIG. 66.

EXAMPLE.—In the right-angled triangle ABC , Fig. 67, right-angled at C , $AB = 308$ feet and $BC = 234$ feet; to find the other parts.

SOLUTION.— $\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{234}{308} = .75974$. (Art. 594.)

The angle whose sine is .75974 is $49^\circ 26\frac{1}{2}'$, nearly, = angle A . Referring to the table of Sines, the sine of $49^\circ 26'$ is .75965, and the sine of $49^\circ 27'$ is .75984. Since the value obtained, .75974, lies nearly half way between these values, the angle lies midway between the above angles, and is $49^\circ 26\frac{1}{2}'$. Angle $B = 90^\circ - 49^\circ 26\frac{1}{2}' = 40^\circ 33\frac{1}{2}'$.

To find AC , use relation IV, Art. 609.

Side adjacent $A = \text{side opposite} \times \cot A$, or $AC = 234 \times .85586 = 200.27$ feet.

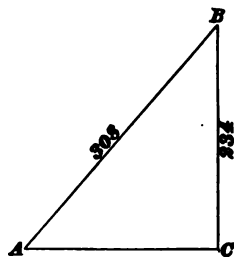


FIG. 67.

$$\text{Ans. } \begin{cases} \text{Angle } A = 49^\circ 26\frac{1}{2}'. \\ \text{Angle } B = 40^\circ 33\frac{1}{2}'. \\ AC = 200.27 \text{ ft.} \end{cases}$$

EXAMPLES FOR PRACTICE.

613. Solve the following examples:

1. In a right triangle ABC , right-angled at C , the hypotenuse $AB = 40$ inches and angle $A = 28^\circ 14' 14''$. Solve the triangle.

$$\text{Ans. } \begin{cases} \text{Angle } B = 61^\circ 45' 46''. \\ AC = 35.24 \text{ in.} \\ BC = 18.92 \text{ in.} \end{cases}$$

2. In a right triangle ABC , right-angled at C , the side $BC = 10$ feet 4 inches. If angle $A = 26^\circ 59' 6''$, what do the other parts equal?

$$\text{Ans. } \begin{cases} \text{Angle } B = 63^\circ 0' 54''. \\ AB = 22 \text{ ft. } 9\frac{1}{2} \text{ in., nearly.} \\ AC = 20 \text{ ft. } 3\frac{1}{4} \text{ in., nearly.} \end{cases}$$

3. In a right triangle ABC , right-angled at C , the hypotenuse $AB = 60$ feet and the side $AC = 22$ feet. Solve the triangle.

$$\text{Ans. } \begin{cases} \text{Angle } A = 68^\circ 29' 22.2''. \\ \text{Angle } B = 21^\circ 30' 37.8''. \\ BC = 55.82 \text{ ft.} \end{cases}$$

4. In a right triangle ABC , right-angled at C , side $AC = .364$ foot and side $BC = .216$ foot. Solve the triangle.

$$\text{Ans. } \begin{cases} \text{Angle } A = 30^\circ 41' 7.5''. \\ \text{Angle } B = 59^\circ 18' 52.5''. \\ AB = .423 \text{ ft.} \end{cases}$$

OBLIQUE-ANGLED PLANE TRIANGLES.

614. We will give here the method of solving *any* oblique-angled plane triangle, when (1) *two sides and an angle opposite one of them* are given, and (2) when *two angles and a side opposite one of them* are given. Let the student bear in mind, however, that he may use this method of solution only when he knows the general form of the triangle of which he desires the values of some of the parts. This is necessary because in some cases *two solutions* are possible, resulting in the determinations of triangles of quite different forms. We do not think it necessary to go so deeply into the explanation of this point that the student can detect the cases in which two solutions are possible.

615. The solution of the triangle depends upon the following principle: *In any triangle, the sides are proportional to the sines of the opposite angles.* Thus, referring to Fig. 68, the following proportions are true:

$$\begin{aligned} a : b &= \sin A : \sin B. \\ a : c &= \sin A : \sin C. \\ b : c &= \sin B : \sin C. \end{aligned}$$

The method of solving the triangle is shown in the following examples:

616. Case I.—*Two sides and an angle opposite one of them are given :*

EXAMPLE.—In the triangle ABC , Fig. 68, having given the side $a = 1,686$ feet, the side $b = 960$ feet and the angle $A = 33^\circ 35'$; to find the angle B .

SOLUTION.—We have

$$a : b = \sin A : \sin B.$$

Substituting the known values, we have

$$1,686 : 960 = \sin \text{ of } 33^\circ 35' : \sin B.$$

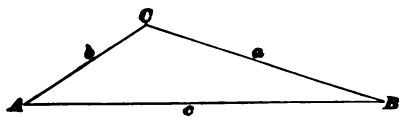


FIG. 68.

From the table of Natural Sines we get .55315 as the sine of $33^\circ 35'$. Substituting this value in the proportion and solving, we find that .31496 is the sine of the angle B , which, by consulting the table of Natural Sines, we find to correspond nearly with the angle $18^\circ 22'$. Ans.

617. Case II.—*Two angles and a side opposite one of them are given :*

EXAMPLE.—In the triangle ABC , given the angle $A = 33^\circ 35'$, the angle $B = 18^\circ 22'$, and the side $a = 1,686$ feet; to find the side b .

SOLUTION.—We have $\sin A : \sin B = a : b$. Substituting known values, we get

$$.55315 : .31509 = 1686 : b.$$

Solving the proportion, we find $b = 960$ feet. Ans.



LOGARITHMS.

EXPONENTS.

618. By the use of logarithms, the processes of multiplication, division, involution, and evolution, are greatly shortened, and some operations may be performed that would be impossible without them. Ordinary logarithms cannot be applied to addition and subtraction.

619. The **logarithm** of a number is that *exponent* by which some fixed number, called the **base**, must be affected in order to equal the number. Any number may be taken as the base. Suppose we choose 4. Then, the logarithm of 16 is 2, because 2 is the exponent by which 4 (the base) must be affected in order to equal 16, since $4^2 = 16$. In this case, instead of reading 4^2 as 4 square, read it 4 exponent 2. With the same base, the logarithms of 64 and 8 would be 3 and 1.5, respectively, since $4^3 = 64$, and $4^{1.5} = 8$. In these cases, as in the preceding, read 4^3 and $4^{1.5}$ as 4 exponent 3, and 4 exponent 1.5, respectively.

620. Although any number *can* be used as a base, and a table of logarithms calculated, but two numbers have ever been employed. For all arithmetical operations (except addition and subtraction), the logarithms used are called the **Briggs or common** logarithms, and the base used is 10. In abstract mathematical analysis, the logarithms used are variously called **hyperbolic, Napierian, or natural** logarithms, and the base is $2.718281828+$. The common logarithm of any number may be converted into a Napierian logarithm by multiplying the common logarithm by $2.30258508+$, which is usually abbreviated to 2.3026, and sometimes to 2.3. Only the common system of logarithms will be considered in this Course.

621. Since, in the common system, the base is 10, it follows that, since $10^1 = 10$; $10^2 = 100$; $10^3 = 1,000$, etc., the logarithm (exponent) of 10 is 1; of 100 is 2; of 1,000 is 3, etc. For the sake of brevity in writing, the words "logarithm of," are abbreviated to "log." Thus, instead of writing logarithm of $100 = 2$, write $\log 100 = 2$. When speaking, however, the words for which "log" stands should always be pronounced in full.

622. From the above it will be seen (see Arts. 503 and 564) that, when the base is 10,

Since $10^0 = 1$, the exponent $0 = \log 1$;

" $10^1 = 10$, " " $1 = \log 10$;

" $10^2 = 100$, " " $2 = \log 100$;

" $10^3 = 1,000$, " " $3 = \log 1,000$, etc.

Also,

Since $10^{-1} = \frac{1}{10} = .1$ the exponent $-1 = \log .1$;

" $10^{-2} = \frac{1}{100} = .01$, " " $-2 = \log .01$;

" $10^{-3} = \frac{1}{1000} = .001$, " " $-3 = \log .001$, etc.

From this it will be seen that the logarithms of exact powers of 10 and of decimals like .1, .01, and .001 are the whole numbers 1, 2, 3, etc., and -1 , -2 , -3 , etc., respectively. *Only numbers consisting of 1 and one or more ciphers have whole numbers for logarithms.*

623. Now, it is evident that to produce a number between 1 and 10, the exponent of 10 must be a fraction; to produce a number between 10 and 100, it must be 1 plus a fraction; to produce a number between 100 and 1,000, it must be 2 plus a fraction, etc. Hence, the logarithm of any number between 1 and 10 is a fraction; of any number between 10 and 100, 1 plus a fraction; of any number between 100 and 1,000, 2 plus a fraction, etc. A logarithm, therefore, usually consists of two parts, a whole number, called the **characteristic**, and a fraction, called the **mantissa**. The mantissa is always expressed as a decimal. For example, to produce 20, 10 must have an exponent of *approximately* 1.30103, or $10^{1.30103} = 20$, very nearly, the degree of exactness depending upon the number of decimal

places used. Hence, $\log 20 = 1.30103$, 1 being the characteristic and .30103 the mantissa.

624. Referring to the second part of the table, Art. **622**, it is clear that the logarithms of all numbers less than 1 are negative, the logarithms of those between 1 and .1 being -1 plus a fraction. For, since $\log .1 = -1$, the logarithms of .2, .3, etc. (which are all greater than .1, but less than 1), must be greater than -1 ; i. e., they must equal -1 plus a fraction. For the same reason, to produce a number between .1 and .01, the logarithm (exponent of 10) would be equal to -2 plus a fraction, and for a number between .01 and .001, it would be equal to -3 plus a fraction. Hence, the logarithm of any number between 1 and .1 has a negative characteristic of 1, and a positive mantissa; of a number between .1 and .01, a negative characteristic of 2, and a positive mantissa; of a number between .01 and .001, a negative characteristic of 3, and a positive mantissa; of a number between .001 and .0001, a negative characteristic of 4, and a positive mantissa, etc. *The negative characteristics are distinguished from the positive by the $-$ sign written over the characteristic.* Thus, $\bar{3}$ indicates that 3 is negative.

It must be remembered that in all cases the mantissa is positive. Thus, the logarithm 1.30103 means $+1 + .30103$, and the logarithm $\bar{1}.30103$ means $-1 + .30103$. Were the minus sign written in front of the characteristic, it would indicate that the entire logarithm was negative. Thus, $-1.30103 = -1 - .30103$.

625. Rules for Characteristic.—From Art. **624**, it follows that:

I. *For a number greater than 1 the characteristic is one less than the number of integral places in the number.*

By "integral places" is meant the figures (including ciphers) to the left of the decimal point.

II. *For a number wholly decimal, the characteristic is negative, and is numerically one greater than the number of*

ciphers between the decimal point and the first digit of the decimal.

For example, the characteristics of the logarithms of 256, 31.24, 7.53, and 1,728.0036, are 2, 1, 0, and 3, respectively, or *one less* than the number of integral places in each case; the characteristics of the logarithms of .0005, .0674, and .50072, are $\bar{4}$, $\bar{2}$, and $\bar{1}$, respectively, or numerically *one greater* than the number of ciphers immediately following the decimal point. It will be noticed that in the last number there are no ciphers, and the characteristic is $0 + \bar{1} = \bar{1}$.

THE LOGARITHMIC TABLE.

TO FIND THE LOGARITHM OF A NUMBER.

626. To aid in obtaining the mantissas of logarithms, *tables of logarithms* have been calculated, some of which are very elaborate and convenient. In the Table of Logarithms, the mantissas of the logarithms of numbers, from 1 to 9,999, are given to five places of decimals, and the mantissas of logarithms of larger numbers can be found by interpolation. The table contains the *mantissas only*, and the characteristics may be easily found by the rules of Art. 625.

The table depends upon the principle, which will be explained later, that all numbers having the same figures in the same order, have their mantissas alike, without regard to the position of the decimal point, which affects the characteristic only. To illustrate, if $\log 206 = 2.31387$, then,

$$\begin{array}{ll} \log 20.6 = 1.31387. & \log .206 = \bar{1}.31387. \\ \log 2.06 = .31387. & \log .0206 = \bar{2}.31387, \text{ etc.} \end{array}$$

627. To find the logarithm of a number not having more than four figures:

Rule.—Find the first three significant figures of the number whose logarithm is desired, in the left-hand column; find the fourth figure in the column at the top (or bottom) of the page, and in the column under (or above) this figure, and

opposite the first three figures previously found will be the mantissa or decimal part of the logarithm. The characteristic being found as described in Art. 625, write it at the left of the mantissa, and the resulting expression will be the logarithm of the required number.

628. EXAMPLE.—Find the logarithm (a) of 476; (b) of 25.47; (c) of 1.073, and (d) of .06313.

SOLUTION.—(a) In order to economize space, and make the labor of finding the logarithms easier, the first two figures of the mantissa are given only in the column headed 0. The last three figures of the mantissa, opposite 476 in the column headed N (N stands for number), page 9 of the tables, are 761, found in the column headed 0; glancing upwards, we find the first two figures of the mantissa, viz., 67. The characteristic is 2; hence, $\log 476 = 2.67761$. Ans.

NOTE.—Since all numbers in the table are decimal fractions, the decimal point is omitted throughout; this is customary in all tables of logarithms.

(b) To find the logarithm of 25.47, we find the first three figures 254, in the column headed N on page 5, and on the same horizontal line, under the column headed 7 (the fourth figure of the given number), will be found the last three figures of the mantissa, viz., 603. The first two figures are evidently 40, and the characteristic is 1; hence, $\log 25.47 = 1.40603$. Ans.

(c) For 1.073, the last three figures of the mantissa are found in the usual manner, in the column headed 3, opposite 107 in the column headed N on page 2, to be 060. It will be noticed that these figures are printed *060, the star meaning that instead of glancing *upwards* in the column headed 0, and taking 02 for the first two figures, we must glance *down* and take the two figures opposite the number 108, in the left-hand column, i. e., 03. The characteristic being 0, $\log 1.073 = 0.03060$, or, more simply, .03060.

(d) For .06313, the last three figures of the mantissa are found opposite 631, in column headed 3 on page 12, to be 024. In this case, the first two figures occur in the same row, and are 80. Since the characteristic is $\bar{2}$, $\log .06313 = \bar{2}.80024$. Ans.

629. If the original number contains but one digit (a cipher is not a digit), annex mentally two ciphers to the right of the digit; if the number contains but two digits (with no ciphers between, as in 4008), annex mentally one cipher on the right, before seeking the mantissa. Thus, if the logarithm of 7 is wanted, seek the mantissa for 700, which is .84510; or, if the logarithm of 48 is wanted, seek the mantissa for

480, which is .68124. Or, find the mantissas of logarithms of numbers between 0 and 100, on the first page of the tables.

The process of finding the logarithm of a number from the table is technically called **taking out the logarithm**.

630. To take out the logarithm of a number consisting of more than four figures, it is inexpedient to use more than five figures of the number, when using five-place logarithms (the logarithms given in the accompanying table are five-place). Hence, if the number consists of more than five figures, and the sixth figure is less than 5, replace all figures after the fifth with ciphers; if the sixth figure is 5 or more, increase the fifth figure by one, and replace the remaining figures with ciphers. Thus, if the number is 31,415,926, find the logarithm of 31,416,000; if 31,415,426, find the logarithm of 31,415,000.

631. EXAMPLE.—Find $\log 31,416$.

SOLUTION.—Find the mantissa of the logarithm of the first four figures, as explained above. This is, in the present case, .49707 (see page 6). Now, subtract the number in the column headed 1, opposite 314 (the first three figures of the given number), from the next greater consecutive number, in this case 721, in the column headed 2. $721 - 707 = 14$; this number is called the **difference**. At the extreme right of the page will be found a secondary table headed P. P., and at the top of one of these columns, in this table, in bold-face type, will be found the difference. It will be noticed that each column is divided into two parts by a vertical line, and that the figures on the left of this line run in sequence from 1 to 9. Considering the difference column headed 14, we see opposite the number 6 (6 is the last or fifth figure of the number whose logarithm we are taking out) the number 8.4, and we add this number to the mantissa found above, disregarding the decimal point in the mantissa, obtaining $49707 + 8.4 = 49715.4$. Now, since 4 is less than 5, we reject it, and obtain for our complete mantissa .49715. Since the characteristic of the logarithm of 31,416 is 4, $\log 31,416 = 4.49715$. Ans.

632. EXAMPLE.—Find $\log 380.93$.

SOLUTION.—Proceeding in exactly the same manner as above, the mantissa for 3,809 is 58081 (the star directs us to take 58 instead of 57 for the first two figures), the next greater mantissa is 58092, found in the column headed 0, opposite 381 in column headed N. The difference is $092 - 081 = 11$. Looking in the section headed P. P., for column

headed 11, we find opposite 8, 3.8; neglecting the .3, since it is less than 5, 3 is the amount to be added to the mantissa of the logarithm of 3800 to form the logarithm of 38003. Hence, $58081 + 3 = 58084$, and since the characteristic is 2, $\log 380.93 = 2.58084$. Ans.

633. EXAMPLE.—Find $\log 1,296,728$.

SOLUTION.—Since this number consists of more than five figures and the sixth figure is less than 5, we find the logarithm of 1,296,700, and call it the logarithm of 1,296,728. The mantissa of $\log 1,296$ is found on page 2 to be 11261. The difference is $294 - 261 = 33$. Looking in the P. P. section for column headed 33, we find opposite 7 on the extreme left, 23.1; neglecting the .1, the amount to be added to the above mantissa is 23. Hence, the mantissa of $\log 1,296,728 = 11261 + 23 = 11284$; since the characteristic is 6, $\log 1,296,728 = 6.11284$. Ans.

634. EXAMPLE.—Find $\log 89.126$.

SOLUTION.— $\log 89.12 = 1.94998$. Difference between this and $\log 89.13 = 1.95002 - 1.94998 = 4$. The P. P. (proportional part) for the fifth figure of the number, 6, is 2.4, or 2.

Hence, $\log 89.126 = 1.94998 + .00002 = 1.95000$. Ans.

635. EXAMPLE.—Find $\log .096725$.

SOLUTION.— $\log .09672 = \overline{2}.98552$. Difference = 4.
P. P. for 5 = 2

Hence, $\log .096725 = \overline{2}.98554$. Ans.

636. To find the logarithm of a number consisting of five or more figures:

Rule.—I. *If the number consists of more than five figures and the sixth figure is 5 or greater, increase the fifth figure by 1, and write ciphers in place of the sixth and remaining figures.*

II. *Find the mantissa corresponding to the logarithm of the first four figures, and subtract this mantissa from the next greater mantissa in the table; the remainder is the difference.*

III. *Find in the secondary table headed P. P. a column headed by the same number as that just found for the difference, and in this column opposite the number corresponding to the fifth figure (or fifth figure increased by 1) of the given number (this figure is always situated at the left of the dividing line of the column) will be found the P. P. (propor-*

tional part) for that number. The P. P. thus found is to be added to the mantissa found in II, as in the preceding examples, and the result is the mantissa of the logarithm of the given number, as nearly as may be found with five-place tables.

EXAMPLES FOR PRACTICE.

637. Find the logarithms of the following numbers:

1. .062.	Ans. $\bar{2}.79239$.
2. 620.	Ans. 2.79239.
3. 21.4.	Ans. 1.33041.
4. .000067.	Ans. $\bar{5}.82607$.
5. 89.42.	Ans. 1.95143.
6. .785398.	Ans. $\bar{1}.89509$.
7. .0010823.	Ans. $\bar{3}.08435$.
8. 10,000.	Ans. 4.
9. 1,923.208.	Ans. 3.28403.
10. 3.00026.	Ans. .47717.

TO FIND A NUMBER WHOSE LOGARITHM IS GIVEN.

638. Rule I.—Consider the mantissa first. Glance along the different columns of the table which are headed 0 until the first two figures of the mantissa are found. Then glance down the same column until the third figure is found (or 1 less than the third figure). Having found the first three figures, glance to the right along the row in which they are situated until the last three figures of the mantissa are found. Then, the number which heads the column in which the last three figures of the mantissa are found is the fourth figure of the required number, and the first three figures lie in the column headed N, and in the same row in which lie the last three figures of the mantissa.

II. If the mantissa cannot be found in the table, find the mantissa which is nearest to, but less than, the given mantissa, and which call the **next less mantissa**. Subtract the next less mantissa from the next greater mantissa in the table to obtain the difference. Also subtract the next less mantissa from the mantissa of the given logarithm, and call the remainder the P. P. Looking in the secondary table headed P. P. for

the column headed by the difference just found, find the number opposite the P. P. just found (or the P. P. corresponding most nearly to that just found); this number is the fifth figure of the required number; the fourth figure will be found at the top of the column containing the next less mantissa, and the first three figures in the column headed N and in the same row which contains the next less mantissa.

III. *Having found the figures of the number as above directed, locate the decimal point by the rules for the characteristic, annexing ciphers to bring the number up to the required number of figures if the characteristic is greater than 4.*

639. EXAMPLE.—Find the number whose logarithm is 3.56867.

SOLUTION.—The first two figures of the mantissa, 56, are found on page 7; glancing down the column, we find the third figure, 8 (in connection with 820), opposite 370 in the N column. Glancing to the right along the row containing 820, the last three figures of the mantissa, 867, are found in the column headed 4; hence, the fourth figure of the required number is 4, and the first three figures are 370, making the figures of the required number 3704. Since the characteristic is 3, there are four figures to the left of the decimal point, and the number whose logarithm is 3.56867 is 3,704. **Ans.**

640. EXAMPLE.—Find the number whose logarithm is 3.56871.

SOLUTION.—The mantissa is not found in the table. The next less mantissa is 56867; the difference between this and the next greater mantissa is $879 - 867 = 12$, and the P. P. is $56871 - 56867 = 4$. Looking in the P. P. section for the column headed 12, we do not find 4, but we do find 3.6 and 4.8. Since 3.6 is nearer 4 than 4.8, we take the number opposite 3.6 for the fifth figure of the required number; this is 3. Hence, the fourth figure is 4; the first three figures 370, and the figures of the number are 37043. The characteristic being 3, the number is 3,704.3. **Ans.**

641. EXAMPLE.—Find the number whose logarithm is 5.95424.

SOLUTION.—The mantissa is found in the column headed 0 on page 18, opposite 900 in the column headed N. Hence, the fourth figure is 0, and the number is 900,000, the characteristic being 5. **Ans.** Had the logarithm been $\bar{5}.95424$, the number would have been .0009.

642. EXAMPLE.—Find the number whose logarithm is .93036.

SOLUTION.—The first three figures of the mantissa, 930, are found in the 0 column opposite 852 in the N column, but since the last two figures of all the mantissas in this row are greater than 36, we must seek

the next less mantissa in the preceding row. We find it to be 93034 (the star directing us to use 93 instead of 92 for the first two figures) in the column headed 8. The difference for this case is $039 - 034 = 5$, and the P. P. is $036 - 034 = 2$. Looking in the P. P. section for the column headed 5, we find the P. P., 2, opposite 4. Hence, the fifth figure is 4; the fourth figure is 8; the first three figures 851, and the number is 8.5184, the characteristic being 0. Ans.

643. EXAMPLE.—Find the number whose logarithm is $\bar{2}.05753$.

SOLUTION.—The next less mantissa is found in column headed 1 opposite 114 in the N column, page 2; hence, the first four figures are 1141. The difference for this case is $767 - 729 = 38$, and the P. P. is $753 - 729 = 24$. Looking in the P. P. section for the column headed 38, we find that 24 falls between 22.8 and 26.6. The difference between 24 and 22.8 is 1.2, and between 24 and 26.6 is 2.6; hence, 24 is nearer 22.8 than it is to 26.6, and 6, opposite 22.8, is the fifth figure of the number. Hence, number whose logarithm is $\bar{2}.05753 = .011416$. Ans.

EXAMPLES FOR PRACTICE.

644. Find the numbers corresponding to the following logarithms:

1. .74429.	Ans. 5.55.
2. 4.38202.	Ans. 24,100.
3. $\bar{1}.84510$.	Ans. .7.
4. 1.84510.	Ans. 70.
5. $\bar{4}.96047$.	Ans. .000913.
6. 3.78942.	Ans. 6,157.7.
7. .50210.	Ans. 3.1776.
8. $\bar{3}.63491$.	Ans. .0043143.
9. $\bar{1}.07619$.	Ans. .11918.
10. $\bar{3}.23417$.	Ans. .0017146.

645. In order to calculate by means of logarithms, a table is absolutely necessary. Hence, for this reason, we do not explain the method of calculating a logarithm. The work involved in calculating even a single logarithm is very great, and no method has yet been demonstrated, of which we are aware, by which the logarithm of a number like 121 can be calculated directly. Moreover, even if the logarithm could be readily obtained, it would be useless without a complete table, such as that which forms a part of this paper, for the reason that after having used it, say to extract a root, the number corresponding to the logarithm of the result could not be found.

MULTIPLICATION BY LOGARITHMS.

646. The principle upon which the process is based may be illustrated as follows: Let X and Y represent two numbers whose logarithms are x and y . To find the logarithm of their product, we have, from the definition of a logarithm,

$$10^x = X, \quad (1)$$

$$\text{and } 10^y = Y. \quad (2)$$

Since both members of (1) may be multiplied by the same quantity without destroying the equality, they evidently may be multiplied by equal quantities like 10^y and Y . Hence, multiplying (1) by (2), member by member,

$$10^x \times 10^y = 10^{x+y} = X Y \text{ (Art. 489);}$$

or, by the definition of a logarithm, $x + y = \log X Y$. But $X Y$ is the product of X and Y , and $x + y$ is the sum of their logarithms; from which it follows that the sum of the logarithms of two members is equal to the logarithm of their product. Hence,

647. To multiply two or more numbers by using logarithms:

Rule.—*Add the logarithms of the several numbers, and the sum will be the logarithm of the product. Find the number corresponding to this logarithm, and the result will be the number sought.*

EXAMPLE.—Multiply 4.38, 5.217, and 83 together.

SOLUTION.—
 Log 4.38 = .64147
 Log 5.217 = .71742
 Log 83 = 1.91908

Adding, $3.27797 = \log (4.38 \times 5.217 \times 83).$

Number corresponding to 3.27797 = 1,896.6. Hence, $4.38 \times 5.217 \times 83 = 1,896.6$, nearly. Ans. By actual multiplication, the product is 1,896.58818, showing that the result obtained by using logarithms was correct to five figures. *

648. When adding logarithms, their *algebraic* sum is always to be found. Hence, if some of their numbers multiplied together are wholly decimal, the algebraic sum of the characteristics will be the characteristic of the product. It must be remembered that the mantissas are always positive.

EXAMPLE.—Multiply 49.82, .00243, 17, and .97 together.

SOLUTION.—Log 49.82 = 1.69740

Log .00243 = $\bar{3}.38561$

Log 17 = 1.23045

Log .97 = $\bar{1}.98677$

Adding, $0.30023 = \log (49.82 \times .00243 \times 17 \times .97)$.

Number corresponding to $0.30023 = 1.9963$. Hence, $49.82 \times .00243 \times 17 \times .97 = 1.9963$. Ans.

In this case the sum of the mantissas was 2.30023. The integral 2 added to the positive characteristics makes their sum $= 2 + 1 + 1 = 4$; sum of negative characteristics $= \bar{3} + \bar{1} = \bar{4}$, whence $4 + (-4) = 0$. If, instead of 17, the number had been .17 in the above example, the logarithm of .17 would have been $\bar{1}.23045$, and the sum of the logarithms would have been 2.30023; the product would then have been .019963.

649. It can now be shown why, as stated in Art. 626, all numbers with figures in the same order have the same mantissa without regard to the decimal point. Thus, suppose it were known that $\log 2.06 = .31387$. Then, $\log 20.6 = \log (2.06 \times 10) = \log 2.06 + \log 10 = .31387 + 1 = 1.31387$. And so it might be proved with the decimal point in any other position.

EXAMPLES FOR PRACTICE.

650. Find the products of the following by the use of logarithms:

- | | |
|-------------------------------------|------------------|
| 1. 100, 32, and 31.64. | Ans. 101,250. |
| 2. 23.1, 59.64, and 7.863. | Ans. 10,833. |
| 3. .00354, .275, and .0198. | Ans. .000019275. |
| 4. 2.763, 59.87, .264, and .001702. | Ans. .074328. |

DIVISION BY LOGARITHMS.

651. As before, let X and Y represent two numbers, whose logarithms are x and y . To find the logarithm of their quotient we have, from the definition of a logarithm,

$$10^x = X, \quad (1)$$

$$\text{and } 10^y = Y. \quad (2)$$

Dividing (1) by (2), $10^{x-y} = \frac{X}{Y}$ (Art. 502), or, by the definition of a logarithm, $x - y = \log \frac{X}{Y}$. But $\frac{X}{Y}$ is the quotient

of $X \div Y$, and $x - y$ is the difference of their logarithms, from which it follows that *the difference between the logarithms of two numbers is equal to the logarithm of their quotient*. Hence,

652. To divide one number by another by means of logarithms:

Rule.—*Subtract the logarithm of the divisor from the logarithm of the dividend, and the result will be the logarithm of the quotient.*

EXAMPLE.—Divide 6,784.2 by 27.42.

SOLUTION.—
 $\text{Log } 6,784.2 = 3.83150$
 $\text{Log } 27.42 = 1.43807$

$\text{difference} = 2.39343 = \log (6,784.2 \div 27.42).$

Number corresponding to 2.39343 = 247.42. Hence, $6,784.2 \div 27.42 = 247.42$. Ans.

653. When subtracting logarithms, their *algebraic* difference is to be found. The operation may sometimes be confusing, because the mantissa is always positive, and the characteristic may be either positive or negative. *When the logarithm to be subtracted is greater than the logarithm from which it is to be taken, or when negative characteristics appear, subtract the mantissa first, and then the characteristic, by changing its sign and adding.* (Art. 475.)

EXAMPLE.—Divide 274.2 by 6,784.2.

SOLUTION.—
 $\text{Log } 274.2 = 2.43807$
 $\text{Log } 6,784.2 = 3.83150$
 $\underline{2.60657}.$

First subtracting the mantissa .83150 gives .60657 for the mantissa of the quotient. In subtracting, 1 had to be taken from the characteristic of the minuend leaving a characteristic of 1. Subtract the characteristic 3 from this, by changing its sign and adding $1 - 3 = \bar{2}$, the characteristic of the quotient. Number corresponding to $\bar{2}.60657 = .040418$. Hence, $274.2 \div 6,784.2 = .040418$. Ans.

EXAMPLE.—Divide .067842 by .002742.

SOLUTION.—
 $\text{Log } .067842 = \bar{2}.83150$
 $\text{Log } .002742 = \bar{3}.43807$
 $\text{difference} = 1.39343.$

Subtracting, $.83150 - .43807 = .39343$ and $-2 + 3 = 1$. Number corresponding to $1.39343 = 24.742$. Hence, $.067842 + .002742 = 24.742$. Ans.

654. The only case that need cause trouble in subtracting is where the logarithm of the minuend has a negative characteristic, or none at all, and a mantissa less than the mantissa of the subtrahend. For example, let it be required to subtract the logarithm 3.74036 from the logarithm $\bar{3}.55145$. The logarithm $\bar{3}.55145$ is equivalent to $-3 + .55145$. Now, if we add both $+1$ and -1 to this logarithm, it will not change its value. Hence, $\bar{3}.55145 = -3 - 1 + 1 + .55145 = \bar{4} + 1.55145$. Therefore, $\bar{3}.55145 - 3.74036 =$

$$\begin{array}{r} \bar{4} + 1.55145 \\ 3 + .74036 \\ \hline \text{difference} = \bar{7} + .81109 = \bar{7}.81109. \end{array}$$

Had the characteristic of the above logarithm been 0 instead of $\bar{3}$, the process would have been exactly the same. Thus, $.55145 = \bar{1} + 1.55145$; hence,

$$\begin{array}{r} \bar{1} + 1.55145 \\ 3 + .74036 \\ \hline \text{difference} = \bar{4} + .81109 = \bar{4}.81109. \end{array}$$

EXAMPLE.—Divide $.02742$ by 67.842 .

SOLUTION.—Log $.02742 = \bar{2}.43807 = \bar{3} + 1.43807$

Log $67.842 = 1.83150 = 1 + .83150$

$$\text{difference} = \bar{4} + .60657 = \bar{4}.60657.$$

Number corresponding to $\bar{4}.60657 = .00040417$. Hence, $.02742 \div 67.842 = .00040417$. Ans.

EXAMPLE.—What is the reciprocal of 3.1416 ?

SOLUTION.—Reciprocal of $3.1416 = \frac{1}{3.1416}$, and $\log \frac{1}{3.1416} = \log 1 - \log 3.1416 = 0 - .49715$. Since $0 = -1 + 1$,

$$\begin{array}{r} \bar{1} + 1.00000 \\ .49715 \\ \hline \text{difference} = \bar{1} + .50285 = \bar{1}.50285. \end{array}$$

Number whose logarithm is $\bar{1}.50285 = .31831$. Ans.

EXAMPLES FOR PRACTICE.

655. Find the quotients of the following by the use of logarithms:

1. $564.35 \div 34.96$.	Ans. 16.143.
2. $9.643 \div 200.04$.	Ans. .048204.
3. $.16071 \div 76.8$.	Ans. .0020926.
4. $.00624 \div 3.096$.	Ans. .0020155.
5. $.000119 \div .0719$.	Ans. .0016551.
6. $1.19 \div 719$.	Ans. .0016551.
7. $1 \div 1,728$.	Ans. .00057870.

INVOLUTION BY LOGARITHMS.

656. If X represents a number whose logarithm is x , we have, from the definition of a logarithm,

$$10^x = X.$$

Raising both numbers to some power, as the n th, the equation becomes, by Art. 567,

$$10^{xn} = X^n.$$

But X^n is the required power of X , and xn is its logarithm, from which it follows that the logarithm of a number multiplied by the exponent of the power to which it is raised is equal to the logarithm of the power. Hence,

657. To raise a number to any power by the use of logarithms:

Rule.—*Multiply the logarithm of the number by the exponent which denotes the power to which the number is to be raised, and the result will be the logarithm of the required power.*

EXAMPLE.—What is the square of (a) 7.92? (b) the cube of 94.7? (c) the 1.6 power of 512, that is, $512^{1.6}$?

SOLUTION.—(a) $\log 7.92 = .89873$; the exponent of the power is 2. Hence, $.89873 \times 2 = 1.79746 = \log 7.92^2$. Number corresponding to 1.79746 = 62.727. Hence, $7.92^2 = 62.727$, nearly. Ans.

(b) $\log 94.7 = 1.97635$; $1.97635 \times 3 = 5.92905 = \log 94.7^3$. Number corresponding to 5.92905 = 849,280. Hence, $94.7^3 = 849,280$, nearly. Ans.

(c) $\log 512^{1.6} = 1.6 \times \log 512 = 1.6 \times 2.70927 = 4.334832$, or 4.33483 (when using five-place logarithms) = $\log 21,619$. Hence, $512^{1.6} = 21,619$, nearly. Ans.

658. If the number is wholly decimal, so that the characteristic is negative, *multiply the two parts of the logarithm separately by the exponent of the number. If, after multiplying the mantissa, the product has a characteristic, add it, algebraically, to the negative characteristic, multiplied by the exponent, and the result will be the negative characteristic of the required power.*

EXAMPLE.—Raise .0751 to the fourth power.

SOLUTION.— $\log .0751^4 = 4 \times \log .0751 = 4 \times \bar{2}.87564$. Multiplying the parts separately, $4 \times \bar{2} = \bar{8}$ and $4 \times .87564 = 3.50256$. Adding the 3 and 8, $3 + (-8) = -5$; therefore, $\log .0751^4 = \bar{5}.50256$. Number corresponding to this = .00003181. Hence, $.0751^4 = .00003181$. Ans.

659. A decimal may be raised to a power whose exponent contains a decimal as follows:

EXAMPLE.—Raise .8 to the 1.21 power.

SOLUTION.— $\log .8^{1.21} = 1.21 \times \bar{1}.90309$. There are several ways of performing the multiplication.

First Method.—Adding the characteristic and mantissa algebraically, the result is $-.09691$. Multiplying this by 1.21 gives $-.1172611$, or $-.11726$, when using 5-place logarithms. To obtain a positive mantissa, add +1 and -1 ; whence, $\log .8^{1.21} = -1 + 1 - .11726 = \bar{1}.88274$.

Second Method.—Multiplying the characteristic and mantissa separately gives $-1.21 + 1.09274$. Adding characteristic and mantissa algebraically gives $-.11726$; then, adding +1 and -1 , $\log .8^{1.21} = \bar{1}.88274$.

Third Method.—Multiplying the characteristic and mantissa separately gives $-1.21 + 1.09274$. Adding the decimal part of the characteristic to the mantissa gives $-1 + (-.21 + 1.09274) = \bar{1}.88274 = \log .8^{1.21}$. The number corresponding to the logarithm $\bar{1}.88274 = .76338$. Ans.

Any one of the above three methods may be used, but we recommend the first or the third. The third is the most elegant, and saves figures, but requires the exercise of more caution than the first method does. Below will be found the entire work of multiplication for both $.8^{1.21}$ and $.8^{.21}$.

1.90309	1.90309
1.21	.21
90309	90309
180618	180618
90309	+ 1.1896489
1.0927389	- 1 - .21
- 1.21	1.9796489, or 1.97965.
1.8827389, or 1.88274.	

In the second case, the negative decimal obtained by multiplying -1 and $.21$ was greater than the positive decimal obtained by multiplying $.90309$ and $.21$; hence, $+1$ and -1 were added as shown.

EXAMPLES FOR PRACTICE.

660. Find the values of the following by logarithms:

- | | |
|--------------------|------------------|
| 1. $1,728^3$. | Ans. 2,985,900. |
| 2. $2.49^{1.34}$. | Ans. 3.0995. |
| 3. $32.16^{.43}$. | Ans. 4.2961. |
| 4. $.64^4$. | Ans. .16777. |
| 5. $.64^4$. | Ans. .8365. |
| 6. $.0241^{2.3}$. | Ans. .000029489. |

EVOLUTION BY LOGARITHMS.

661. If X represents a number whose logarithm is x , we have, from the definition of a logarithm,

$$10^x = X.$$

Extracting some root of both members, as the n th, the equation becomes, by Art. 568,

$$10^{\frac{x}{n}} = \sqrt[n]{X}.$$

But $\sqrt[n]{X}$ is the required root of X , and $\frac{x}{n}$ is its logarithm, from which it follows that the logarithm of a number, divided by the index of the root to be extracted, is equal to the logarithm of the root. Hence,

662. To extract any root of a number by means of logarithms:

Rule.—*Divide the logarithm of the number by the index of the root; the result will be the logarithm of the root.*

EXAMPLE.—Extract (a) the square root of 77,851; (b) the cube root of 698,970; (c) the 2.4 root of 8,964,300.

SOLUTION.—(a) Log 77,851 = 4.89127; the index of the root is 2; hence, $\log \sqrt{77,851} = 4.89127 \div 2 = 2.44564$; number corresponding to this = 279.02. Hence, $\sqrt{77,851} = 279.02$, nearly. Ans.

(b) Log $\sqrt[3]{698,970} = 5.84446 \div 3 = 1.94815 = \log 88.746$; or, $\sqrt[3]{698,970} = 88.747$, nearly. Ans.

(c) $\text{Log } \sqrt[4]{8,964,900} = 6.95251 + 2.4 = 2.89688 = \log 788.64$; or,
 $\sqrt[4]{8,964,900} = 788.64$, nearly. Ans.

663. If it is required to extract a root of a number wholly decimal, and the negative characteristic will not exactly contain the index of the root, without a remainder, proceed as follows:

Separate the two parts of the logarithm; add as many units (or parts of a unit) to the negative characteristic as will make it exactly contain the index of the root. Add the same number to the mantissa, and divide both parts by the index. The result will be the characteristic and mantissa of the root.

EXAMPLE.—Extract the cube root of .0003181.

$$\text{SOLUTION.}—\text{Log } \sqrt[3]{.0003181} = \frac{\log .0003181}{3} = \frac{\overline{4}.50256}{3}.$$

$$(\overline{4} + \overline{2} = \overline{6}) + (2 + .50256 = 2.50256).$$

$$(\overline{6} + 3 = \overline{3}) + (2.50256 + 3 = .83419); \text{ or,}$$

$$\log \sqrt[3]{.0003181} = \overline{3}.83419 = \log .068263.$$

$$\text{Hence, } \sqrt[3]{.0003181} = .068263. \text{ Ans.}$$

EXAMPLE.—Find the value of $\sqrt[4]{.0003181}$.

$$\text{SOLUTION.}—\text{Log } \sqrt[4]{.0003181} = \frac{\log .0003181}{4} = \frac{\overline{4}.50256}{4}.$$

If $-.23$ be added to the characteristic, it will contain 1.41 exactly 3 times. Hence,

$$[-4 + (-.23) = -4.23] + [.23 + .50256 = .73256].$$

$$(-4.23 + 1.41 = \overline{3}) + (.73256 + 1.41 = .51955); \text{ or,}$$

$$\log \sqrt[4]{.0003181} = \overline{3}.51955 = \log .0033079.$$

$$\text{Hence, } \sqrt[4]{.0003181} = .0033079. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

664. Find the values of the following by logarithms:

- | | |
|----------------------------|--------------|
| 1. $\sqrt[4]{906.8}$. | Ans. 5.4876. |
| 2. $\sqrt[4]{11}$. | Ans. 1.6154. |
| 3. $.0497^{\frac{1}{2}}$. | Ans. .36766. |
| 4. $.128^{\frac{1}{3}}$. | Ans. .7099. |
| 5. $\sqrt[4]{.0227}$. | Ans. .21999. |
| 6. $\sqrt[4]{.756}$. | Ans. .62738. |

665. EXAMPLE.—Solve this expression by logarithms:

$$\frac{497 \times .0181 \times 762}{3,900 \times .6517} = ?$$

SOLUTION.—

$$\text{Log } 497 = 2.69636$$

$$\text{Log } .0181 = \bar{2}.25768$$

$$\text{Log } 762 = 2.88195$$

$$\text{Log product} = 3.88599$$

$$\text{Log } 3,900 = 3.51851$$

$$\text{Log } .6517 = \bar{1}.81405$$

$$\text{Log product} = 3.33256$$

$$3.88599 - 3.33256 = .50343 = \log 3.1874.$$

Hence,
$$\frac{497 \times .0181 \times 762}{3,900 \times .6517} = 3.1874. \text{ Ans.}$$

EXAMPLE.—Solve, $\sqrt[3]{\frac{504,203 \times 507}{1.75 \times 71.4 \times 87}}$ by logarithms

SOLUTION.—

$$\text{Log } 504,203 = 5.70260$$

$$\text{Log } 507 = 2.70501$$

$$\text{Log product} = 8.40761$$

$$\text{Log } 1.75 = .24304$$

$$\text{Log } 71.4 = 1.85370$$

$$\text{Log } 87 = 1.93952$$

$$\text{Log product} = 4.03626$$

$$\frac{8.40761 - 4.03626}{3} = 1.45712 = \log 28.65.$$

Hence,
$$\sqrt[3]{\frac{504,203 \times 507}{1.75 \times 71.4 \times 87}} = 28.65. \text{ Ans.}$$

666. Logarithms can often be applied to the solution of equations.

EXAMPLE.—Solve the equation $2.43x^5 = \sqrt[3]{.0648}$.

SOLUTION.— $2.43x^5 = \sqrt[3]{.0648}$.

Dividing by 2.43,
$$x^5 = \frac{\sqrt[3]{.0648}}{2.43}.$$

Taking the logarithm of both numbers,

$$5 \times \log x = \frac{\log .0648}{3} - \log 2.43;$$

$$\begin{aligned} \text{or, } 5 \log x &= \frac{2.81158}{3} - .38561. \\ &= 1.80193 - .38561. \\ &= 1.41632. \end{aligned}$$

Dividing by 5, $\log x = 1.88326;$
whence, $x = .7643. \text{ Ans.}$

EXAMPLES FOR PRACTICE.

667. Find the values of the following:

$$1. \frac{89 \times 753 \times .0097}{36,709 \times .08497}. \quad \text{Ans. .20840.}$$

$$2. \sqrt[3]{\frac{7,932 \times .00657 \times .80464}{.03274 \times .6428}}. \quad \text{Ans. 12.583.}$$

$$3. \sqrt[7]{\frac{.03271^3 \times 53.429 \times .77542^5}{32.769 \times .000371^4}}. \quad \text{Ans. 33.035.}$$

Find the value of x in the following:

$$4. 5x^7 = \frac{129.4 \times .71}{90}. \quad \text{Ans. } x = .93237.$$

$$5. 38x^{.43} = \frac{129.4 \times .71^3}{\sqrt[4]{80}}. \quad \text{Ans. } x = .063133.$$

NOTICE.

There is a break in the continuity of the article numbers, figure numbers, etc., as well as in the question numbers, between the section on Logarithms and the section on Elementary Mechanics. This does not affect the subject matter, which is published in full, and in regular order.



ELEMENTARY MECHANICS.

MATTER AND ITS PROPERTIES.

DEFINITIONS.

828. **Matter** is anything that occupies space. It is the substance of which all bodies are composed. Matter is composed of *molecules* and *atoms*.

829. A **molecule** is the smallest portion of matter that can exist without changing its nature.

830. An **atom** is an indivisible portion of matter.

Atoms unite to form molecules, and a collection of molecules form a mass or body.

A drop of water may be divided and subdivided, until each particle is so small that it can only be seen by the most powerful microscope, but each particle will still be water. Now, imagine the division to be carried on still farther until a limit is reached beyond which it is impossible to go without changing the nature of the particle. The particle of water is now so small that, if it be divided again, it will cease to be water, and will be something else; we call this particle a *molecule*.

If a molecule of water be divided, it will yield two atoms of hydrogen gas, and one of oxygen gas. If a molecule of sulphuric acid be divided, it will yield two atoms of hydrogen, one of sulphur, and four of oxygen.

It has been calculated that the diameter of a molecule is larger than $\frac{1}{100000000}$ of an inch, and smaller than $\frac{1}{5000000000}$ of an inch.

831. **Bodies** are composed of collections of molecules. Matter exists in three conditions or forms: *solid*, *liquid*, and *gaseous*.

832. A **solid body** is one whose molecules change their relative positions with great difficulty; as iron, wood, stone, etc.

833. A **liquid body** is one whose molecules tend to change their relative positions easily. Liquids readily adapt themselves to the vessel which contains them, and their upper surface always tends to become perfectly level. Water, mercury, molasses, etc., are liquids.

834. A **gaseous body**, or gas, is one whose molecules tend to separate from one another; as air, oxygen, hydrogen, etc.

Gaseous bodies are sometimes called **aeriform** (air-like) **bodies**. They are divided into two classes—the so-called “*permanent*” gases, and *vapors*.

835. A **permanent gas** is one which remains a gas at ordinary temperatures and pressures.

836. A **vapor** is a body which at ordinary temperatures is a liquid or solid, but, when heat is applied, becomes a gas, as steam.

One body may be in all three states; as, for example, mercury, which at ordinary temperatures is a liquid, becomes a solid (freezes) at 40° below zero, and a vapor (gas) at 600° above zero. By means of great cold, all gases, even hydrogen, have been liquefied, and some solidified.

By means of heat, all solids have been liquefied and a great many vaporized. It is probable that if we had the means of producing sufficiently great extremes of heat and cold, all solids might be converted into gases, and all gases into solids.

837. Every portion of matter possesses certain qualities called *properties*. Properties of matter are divided into two classes: *general* and *special*.

838. **General properties of matter** are those which are common to all bodies. They are as follows: *Extension, impenetrability, weight, indestructibility, inertia, mobility, divisibility, porosity, compressibility, expansibility, and elasticity.*

839. Special properties are those which are not possessed by all bodies. Some of the most important are as follows: *Hardness, tenacity, brittleness, malleability, and ductility.*

840. Extension is the property of occupying space. Since all bodies must occupy space, it follows that extension is a general property.

841. By impenetrability we mean that no two bodies can occupy exactly the same space at the same time.

842. Weight is the measure of the earth's attraction upon a body. All bodies have weight. In former times it was supposed that gases had no weight, since, if unconfined, they tend to move away from the earth; but, nevertheless, they will finally reach a point beyond which they cannot go, being held in suspension by the earth's attraction. Weight is measured by comparing it with a standard. The standard is a bar of platinum owned and kept by the government; it weighs one pound.

843. Inertia means that a body cannot put itself in motion nor bring itself to rest. To do so, it must be acted upon by some force.

844. Mobility means that a body can be changed in position by some force acting upon it.

845. Divisibility is that property of matter which indicates that a body may be separated into parts.

846. Porosity is that property of matter which indicates that there is space between the molecules of a body. Molecules of bodies are supposed to be spherical, and, hence, there is space between them, as there would be between peaches in a basket. The molecules of water are larger than those of salt; so that when salt is dissolved in water, its molecules wedge themselves between the molecules of the water, and, unless too much salt is added, the water will occupy no more space than it did before. This does not prove that water is penetrable, for the molecules of salt occupy the space that the molecules of water did not.

Water has been forced through iron by pressure, thus proving that iron is porous.

847. Compressibility is that property of matter which indicates that the molecules of a body may be crowded nearer together, so as to occupy a smaller space.

848. Expansibility is that property of matter which indicates that the molecules of a body may be forced apart, so as to occupy a greater space.

849. Elasticity is that property of matter which indicates that if a body be distorted within certain limits, it will resume its original form when the distorting force is removed. Glass, ivory, and steel are very elastic.

850. Indestructibility indicates that matter can never be destroyed. A body may undergo thousands of changes; be resolved into its molecules, and its molecules into atoms, which may unite with other atoms to form other molecules and bodies, which may be entirely different from the original body, but the same number of atoms remains. The whole number of atoms in the universe is exactly the same now as it was millions of years ago, and will always be the same. *Matter is indestructible.*

851. Hardness is that property of matter which indicates that some bodies may scratch other bodies. Fluids and gases do not possess hardness. The diamond is the hardest of all substances.

852. Tenacity is that property of matter which indicates that some bodies resist a force tending to pull them apart. Steel is very tenacious.

853. Brittleness is that property of matter which indicates that some bodies are easily broken; as glass, crockery, etc.

854. Malleability is that property of matter which indicates that some bodies may be hammered or rolled into sheets. Gold is the most malleable of all substances.

855. Ductility is that property of matter which indicates that some bodies may be drawn into wire. Platinum is the most ductile of substances.

856. Mechanics is that science which treats of the action of forces upon bodies, and the effects which they produce; it treats of the laws which govern the movement and equilibrium of bodies, and shows how they may be utilized.

MOTION AND REST.

VELOCITY.

857. Motion is the opposite of rest, and indicates a changing of position in relation to some object. If a large stone is rolled down hill, it is in motion in relation to the hill.

If a person is on a railway train, and walks in the opposite direction from that in which the train is moving, and with the same speed, he will be in motion as regards the train, but at rest with respect to the earth, since, until he gets to the end of the train, he will be directly over the spot at which he was when he started to walk.

858. The path of a body in motion is the line described by its *central point*. No matter how irregular the shape of the body may be, nor how many turns and twists it may make; the line which indicates the direction of the center of the body for every instant that it was in motion, is the path of the body.

859. Velocity is rate of motion. It is measured by a unit of space passed over in a unit of time. When equal spaces are passed over in equal times, the velocity is said to be **uniform**. In all other cases it is **variable**.

If the fly-wheel of an engine keeps up a constant speed of a certain number of revolutions per minute, the velocity of any point on the wheel is uniform. A railway train having a constant speed of 40 miles per hour, moves 40 miles every hour, or $\frac{40}{60} = \frac{2}{3}$ of a mile every minute; and, since equal spaces are passed over in equal times, the velocity is uniform.

Let S = the length of space passed over uniformly;
 t = the time occupied in passing over the space S ;
 V = the velocity.

Then, the velocity V must equal the space S , divided by the time t , or

$$V = \frac{S}{t}. \quad (7.)$$

Also, the space S must equal the velocity V , multiplied by the time, or

$$S = Vt. \quad (8.)$$

The time t must equal the space S , divided by the velocity, or

$$t = \frac{S}{V}. \quad (9.)$$

860. Unless stated otherwise, the *space passed over* will be the *length of the path of the body*, and will be measured in *feet* and decimals of a foot, and, unless otherwise stated, the *time* will be measured in *seconds*.

When these *units* are used, the *velocity will be in feet per second*, which means that the center of the body passed over a certain *number of feet every second*, and the *unit* will be *one foot in one second*.

EXAMPLE.—The velocity of sound in still air is 1,092 feet per second. If I see the flash of a cannon when it is fired, but do not hear the report until 5 seconds afterwards, how far away is the cannon?

SOLUTION.— $S = Vt = 1,092 \times 5 = 5,460$ feet. Ans.

EXAMPLE.—The velocity of light is 186,000 miles a second. If the average distance from the earth to the sun is 93,000,000 miles, how long does it take for a beam of light to reach the earth from the sun?

SOLUTION.— $t = \frac{S}{V} = \frac{93,000,000}{186,000} = 500$ seconds, or 8 minutes 20 seconds. Ans.

EXAMPLE.—If a body passes over a space of 4,800 feet uniformly in 8 minutes, what is its velocity in feet per second?

SOLUTION.—8 minutes = 480 seconds. $V = \frac{S}{t} = \frac{4,800}{480} = 10$. Hence, the velocity is 10 feet per second. Ans.

In examples concerning *work* the *unit of velocity* is usually taken as *one foot in one minute*.

The *unit of time* may be a *second, minute, hour, day, or year*. The *unit of space* may be *feet, miles, the earth's radius*, or the *distance from the earth to the sun*, according to the conditions of the example. The larger units are used only in astronomy.

EXAMPLE.—The distance from the earth to the moon is about 60 times the radius of the earth; how many miles is it from here to the moon?

SOLUTION.—The radius of the earth is nearly 4,000 miles; hence, $4,000 \times 60 = 240,000$ miles, the distance to the moon, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. The piston speed of a steam engine is 10 feet per second; how many miles will the piston travel in one hour? Ans. $6\frac{2}{11}$ mi.
2. If a railroad train travels 70 miles in one hour, what is its velocity in feet per second? Ans. $102\frac{2}{3}$ ft. per sec.
3. A man runs 100 yards in 12 seconds; how long will it take him to run a mile at the same rate? Ans. 3 min. 31.2 sec.
4. The outside diameter of an engine fly-wheel is 13 feet 9 inches. A point on the rim travels 45,000 feet in 5 minutes; what is the velocity in feet per second? Ans. 150 ft. per sec.

FORCE.

THE THREE LAWS OF MOTION.

861. A **force** is that which produces, or tends to produce or destroy, motion. Forces are called by various names, according to the effects which they produce upon a body, as *attraction, repulsion, cohesion, adhesion, accelerating force, retarding force, resisting force*, etc., but all are equivalent to a push or pull, according to the direction in which they act upon a body. That the effect of a force upon a body may be compared with another force, it is necessary that three conditions be fulfilled in regard to both forces; they are as follows:

- (1.) *The point of application, or point at which the force acts upon the body, must be known.*
- (2.) *The direction of the force, or, what is the same thing, the straight line along which the force tends to move the point of application, must be known.*

(3.) *The magnitude or value of the force, when compared with a given standard, must be known.*

862. The unit of magnitude of forces will *always* be taken as *one pound*, in this section on Elementary Mechanics, and all forces will be spoken of as a certain number of pounds.

863. According to the effects which forces produce upon a body, the science of Mechanics is subdivided as follows:

(1.) *Mechanics of Solid Bodies.*

(2.) *Mechanics of Fluids.*

(3.) *Mechanics of Heat, or Thermodynamics.*

Mechanics of Solids is further divided into *Statics* and *Kinetics*, or *Dynamics*, as it is commonly called.

Mechanics of Fluids is further divided into *Mechanics of Air and Gases*, or *Pneumatics*, and *Mechanics of Liquids*. The Mechanics of Liquids is divided into *Hydrostatics* and *Hydrokinetics*; the latter is also called *Hydraulics* and *Hydrodynamics*.

864. Statics treats of the conditions of the equilibrium of bodies. A body is in **equilibrium** under the action of forces, when the forces acting upon the body balance each other.

865. Kinetics, or **Dynamics**, treats of bodies in motion, and the effects which they may produce.

866. Pneumatics treats of the laws of the pressure and of the movement of air and other gaseous bodies.

867. Hydrostatics treats of the equilibrium of liquids.

868. Hydrokinetics (also called *Hydraulics* and *Hydrodynamics*) treats of liquids in motion, and the effects which they may produce.

869. Thermodynamics treats of the mechanical effects of heat upon bodies.

870. The fundamental principles of the relations between force and motion were first stated by Sir Isaac Newton. They are called "Newton's Three Laws of Motion," and are as follows:

(I.) *All bodies continue in a state of rest, or of uniform motion, in a straight line, unless acted upon by some external force that compels a change.*

(II.) *Every motion, or change of motion, is proportional to the acting force, and takes place in the direction of the straight line along which the force acts.*

(III.) *To every action there is always opposed an equal and contrary reaction.*

From the *first law of motion*, it is inferred that a body once set in motion by any force, no matter how small, will move forever in a straight line, and always with the same velocity, unless acted upon by some other force which compels a change. It is not possible to actually verify this law, on account of the earth's attraction for all bodies, but, from astronomical observations, we are certain that the law is true. This law is often called *the law of inertia*.

871. The word **inertia** is so abused that a full understanding of its meaning is necessary. Inertia is not a force, although it is often so called. If a force acts upon a body and puts it in motion, the effect of the force is stored in the body; and a second body, in stopping the first, will receive a blow equal in every respect to the original force, assuming that there has been no resistance of any kind to the motion of the first body.

It is dangerous for a person to jump from a fast-moving train, for the reason that, since his body has the same velocity as the train, it has the same force stored in it that would cause a body of the same weight to take the same velocity as the train, and the effect of a sudden stoppage is the same as the effect of a blow necessary to give the person that velocity. But, by "bracing" himself and jumping in the same direction that the train is moving, and running, he brings himself gradually to rest, and thus reduces the danger. If a body is at rest, it must be acted upon by a force in order to be put in motion, and, no matter how great the force may be, the body cannot be *instantly* put in motion.

The resistance thus offered to being put in motion is commonly, but erroneously, called, the *Resistance of Inertia*. It should be called the *Resistance due to Inertia*.

From the *second law*, it is seen that, if two or more forces act upon a body, their final effect upon the body will be in proportion to their magnitude and to the directions in which they act. Thus, if the wind is blowing due west, with a velocity of 50 miles per hour, and a ball is thrown due north

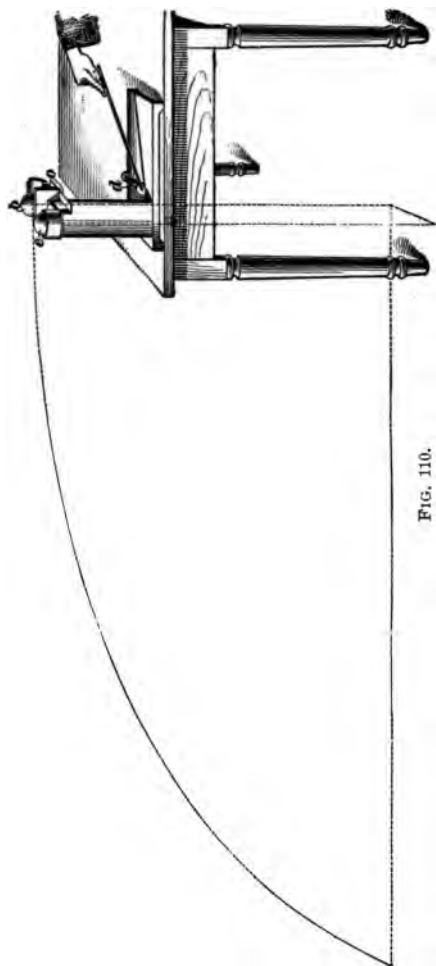


FIG. 110.

with the same velocity, or 50 miles per hour, the wind will carry the ball just as far west as the force of the throw carried it north, and the combined effect will be to cause it to move north-west. The amount of departure from due north will be proportional to the force of the wind, and independent of the velocity due to the force of the throw.

In Fig. 110, a ball *c* is supported in a cup, the bottom of which is attached to the lever *o* in such a manner that a movement of *o* will swing the bottom horizontally and allow the ball to drop. Another ball *b* rests in a horizontal groove that is provided with a slit in the bottom. A swinging arm is actuated by

the spring d in such a manner that, when drawn back as shown and then released, it will strike the lever o and the ball b at the same time. This gives b an impulse in a horizontal direction and swings o so as to allow c to fall.

On trying the experiment, it is found that b follows a path shown by the curved dotted line, and reaches the floor at the same instant as c , which drops vertically. This shows that the force which gave the first ball its horizontal movement, had no effect on the vertical force which compelled both balls to fall to the floor, the vertical force producing the same effect as if the horizontal force had not acted. The second law may also be stated as follows: *A force has the same effect in producing motion, whether it acts upon a body at rest, or in motion, and whether it acts alone or with other forces.*

The *third law* states that action and reaction are equal and opposite. A man cannot lift himself by his boot-straps, for the reason that he presses downwards with the same force that he pulls upwards; the downward reaction equals the upward action, and is opposite to it.

In springing from a boat we must exercise caution, or the reaction will drive the boat from the shore. When we jump from the ground, we tend to push the earth from us, while the earth reacts and pushes us from it.

872. A **force** may be represented by a line; thus, in Fig. 111, let A be the *point of application* of the force; let the length of the line AB represent its *magnitude*, and let the arrow-head indicate the *direction* in which the force acts, then the line AB fulfils the three conditions (see Art. 861), and the force is fully represented.

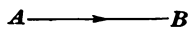


FIG. 111.

THE COMPOSITION OF FORCES.

873. When two forces act upon a body at the same time, but at different angles, their final result may be obtained as follows:

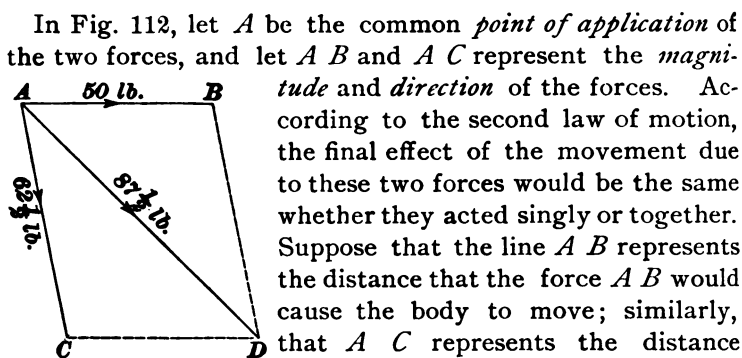


FIG. 112.

In Fig. 112, let A be the common *point of application* of the two forces, and let AB and AC represent the *magnitude* and *direction* of the forces. According to the second law of motion, the final effect of the movement due to these two forces would be the same whether they acted singly or together. Suppose that the line AB represents the distance that the force AB would cause the body to move; similarly, that AC represents the distance which the force AC would cause the body to move when both forces were acting separately. The force AB , acting alone, would carry the body to B ; if the force AC were now to act upon the body, it would carry it along the line BD , parallel to AC , to a point D , at a distance from B equal to AC . Join C and D , then CD is parallel to AB , and $ABDC$ is a parallelogram. Draw the diagonal AD . According to the second law of motion, the body will stop at D , whether the forces act separately or together, but if they act together, the path of the body will be along AD , the diagonal of the parallelogram. Moreover, the length of the line AD represents the *magnitude* of a force which, acting at A in the *direction* AD , would cause the body to move from A to D ; in other words, AD , measured to the same scale as AB and AC , represents, in *magnitude* and *direction*, the combined effect of the two forces AB and AC .

874. This line AD is called the **resultant**. Suppose that the scale used was 50 pounds to the inch; then, if $AB = 50$ pounds, and $AC = 62\frac{1}{2}$ pounds, the length of AB would be $\frac{50}{50} = 1$ inch, and the length of AC would be $\frac{62.5}{50} = 1\frac{1}{4}$ inches. If AD , or the *resultant*, measures $1\frac{3}{4}$ inches, its *magnitude* would be $1\frac{3}{4} \times 50 = 87\frac{1}{2}$ pounds.

Therefore, a force of $87\frac{1}{2}$ pounds acting upon a body at A in the direction AD , will produce the same result as the

combined effects of a force of 50 pounds acting in the direction AB , and a force of $62\frac{1}{2}$ pounds acting in the direction AC .

875. The above method of finding the resulting action of two forces acting upon a body at a common point, is correct, whatever may be their direction and magnitudes. Hence, to find the **resultant** of two forces when their common point of application, their direction and magnitudes are known:

Rule I.—*Assume a point, and draw two lines parallel to the directions of the lines of action of the two forces. With any convenient scale, measure off from the point of intersection (common point of application), distances corresponding to the magnitudes of the respective forces, and complete the parallelogram. From the common point of application, draw the diagonal of the parallelogram; this diagonal will be the resultant, and its direction will be away from the point of application. Its magnitude should be measured with the same scale that was used to measure the two forces.*

This method is called the **graphical method of the parallelogram of forces**.

876. EXPERIMENTAL PROOF.—The principle of the parallelogram of forces is clearly shown in Fig. 113. $ABDC$ is a wooden frame, jointed to allow motion at its four corners. The length AB equals CD ; that of AC equals BD , and the corresponding adjacent sides are in the ratio of two to three. Cords pass over the pulleys M and N , carrying weights W and w , of 90 and 60 pounds. The ratio between the weights equals the ratio of the corresponding adjacent sides. A weight V of 120 pounds is hung from the corner A .

When the frame comes to rest, the sides AB and AC lie in the direction of the cords. These sides AB and AC are accurate graphic representations of the two forces acting upon the point A . It will be found that the diagonal AD is vertical, and twice as long as AC ; hence, since AC represents a force of 60 pounds, AD will represent a force of 2×60 , or 120 pounds.

Thus, we see that the line AD represents the *resultant* of the two forces AB and AC ; in other words, it represents

the resultant of the two weights W and w . This resultant is equal and opposite to the vertical force, which is due to the weight of V .

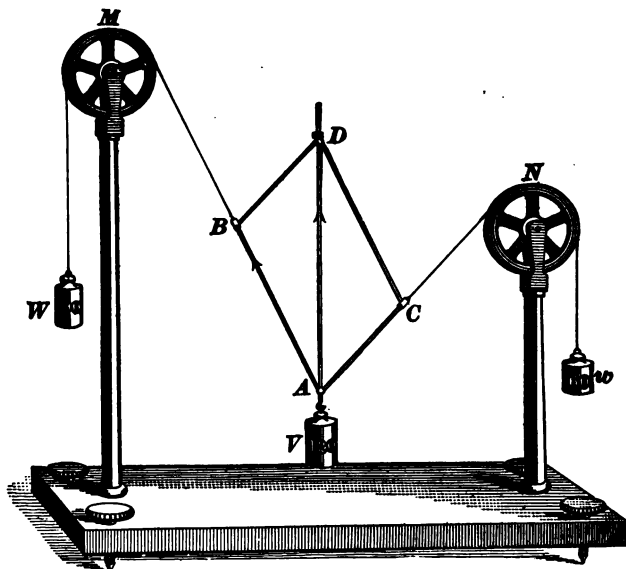


FIG. 118.

Satisfactory results of this kind will be secured when we have the proportion

$$AB : AC = W : w.$$

EXAMPLE.—If two forces act upon a body at a common point, both acting away from the body, and the angle between them is 80° , what is the value of the resultant, the magnitude of the two forces being 60 pounds and 90 pounds, respectively?

SOLUTION.—Draw two indefinite lines having an angle of 80° between them. With any convenient scale, say 10 pounds to the inch, measure off $AB = 60 \div 10 = 6$ inches, and $AC = 90 \div 10 = 9$ inches.

Through B , draw BD parallel to AC , and through C , draw CD parallel to AB , intersecting at D . Then draw AD , and AD will be the *resultant*; its *direction* is towards the point D , as shown by the arrow.

Measuring AD , we find that its length = 11.7 inches. Hence, $11.7 \times 10 = 117$ pounds. Ans.

Caution.—In solving problems by the graphical method, *use as large a scale as possible*. More accurate results are then obtained.

877. The above example might also have been solved

by the method called the **triangle of forces**, which is as follows:

In Fig. 114, suppose that the two forces acted separately, first from A to B , and then from B to D , in the direction of the arrows.

Draw AD ; then AD is the *resultant* of the forces AB and BD , since $BD = AC$; but AD is a side of the triangle ABD . It will also be noticed that the direction of AD is *opposed* to that of AB and BD ; hence, to find the **resultant** of two forces acting upon a body at a common point, by the method of triangle of forces:

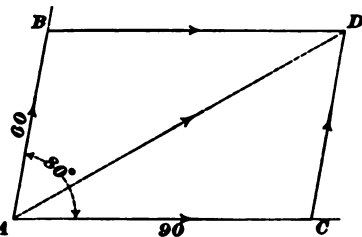


FIG. 114.

Rule II.—*Draw the lines of action of the two forces as if each force acted separately, the lengths of the lines being proportional to the magnitude of the forces. Join the extremities of the two lines by a straight line, and it will be the resultant; its direction will be opposite to that of the two forces.*

NOTE.—When we speak of the resultant being opposed in direction to the other forces around the polygon, we mean that, starting from the point where we began to draw the polygon, and tracing each line in succession, the pencil will have the same general direction around the polygon, as if passing around a circle, from left to right, or from right to left, but that the closing line or resultant must have an *opposite direction*, that is, the two arrow-heads must point towards the point of intersection of the resultant and the last side.

878. When three or more forces act upon a body at a given point, their *resultant* may be found by the following rule:

Rule III.—*Find the resultant of any two forces; treat this resultant as a single force, and combine it with a third force to find a second resultant. Combine this second resultant with a fourth force, to find a third resultant, etc. After all the forces have been thus combined, the last resultant will be the resultant of all of the forces, both in magnitude and direction.*

EXAMPLE.—Find the resultant of all the forces acting on the point O in Fig. 115, the length of the lines being proportional to the magnitude of the forces.

SOLUTION.—Draw OE parallel and equal to AO , and EF parallel and equal to BO , then OF is the resultant of these two forces, and its direction is from O to F , opposed to OE and EF . Treat OF as if OE and EF did not exist, and draw FG parallel and equal to CO ; OG will be the resultant of OF and FG ; but OF is the resultant of OE and EF , hence, OG is the resultant of OE , EF , and FG , and likewise of AO , BO , and CO . The line FG , parallel to CO , could not be drawn from the point O to the right of OE , for in that case it would be opposed in direction to OF ; but FG must have the same direction as OF , in order that the resultant may be opposed to both OF and FG .

For the same reason, draw GL parallel and equal to DO . Join O and L , and OL will be the *resultant* of all the forces AO , BO , CO , and DO (both in magnitude and direction), acting at the point O . If

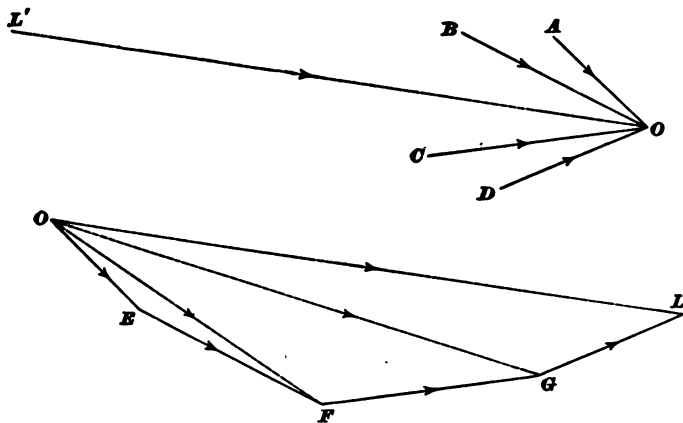


FIG. 115.

$L'O$ were drawn parallel and equal to OL , and having the same direction, it would represent the effect produced on the body by the combined action of the forces AO , BO , CO , and DO .

879. In the last figure, it will be noticed that OE , EF , FG , GL , and LO are sides of a polygon $OEFGL$, in which OL , the resultant, is the closing side, and that its direction is opposed to that of all the other sides. This fact is made use of in what is called the **method of the polygon of forces**.

To find the resultant of several forces acting upon a body at the same point :

Rule IV.—*Through a convenient point on the drawing, draw a line parallel to one of the forces, and having the same*

direction and magnitude. At the end of this line, draw another line parallel to a second force, and having the same direction and magnitude as this second force; at the end of the second line, draw a line parallel and equal in magnitude and direction to a third force. Thus continue until lines have been drawn parallel and equal in magnitude and direction to all of the forces.

The straight line joining the free ends of the first and last lines will be the closing sides of the polygon; mark it opposite in direction to that of the other forces around the polygon, and it will be the resultant of all the forces.

EXAMPLE.—If five forces act upon a body at angles of 60° , 120° , 180° , 240° , and 270° , towards the same point, and their respective magnitudes are 60, 40, 30, 25, and 20 pounds, find the magnitude and direction of their resultant by the method of polygon of forces.

SOLUTION.—From a common point O , Fig. 116, draw the lines of

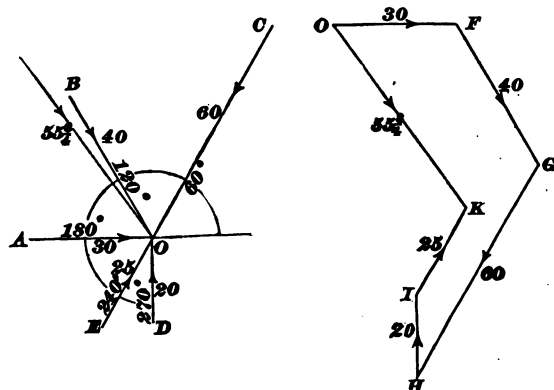


FIG. 116.

action of the forces, making the given angles with a horizontal line through O , and mark them as acting towards O , by means of arrow-heads, as shown. Now, choose some convenient scale, such that the whole figure may be drawn in a space of the required size on the drawing. Choose any one of the forces, as AO , and draw OF parallel to it, and equal in length to 30 pounds on the scale. It must also act in the same direction as OA . At F , draw FG parallel to BO , and equal to 40 pounds. In a similar manner, draw GH , HI , and IK parallel to

CO , DO , and EO , and equal to 60, 20, and 25 pounds, respectively. Join O and K by OK , and OK will be the resultant of the combined action of the five forces; its direction is opposite to that of the other forces around the polygon $OFGH IK$, and its magnitude = $55\frac{1}{2}$ pounds. Ans.

If the resultant OK , in Fig. 116, were to act alone upon the body in the direction shown by the arrow-head, with a force of $55\frac{1}{2}$ pounds, it would produce exactly the same effect upon a body as the combined action of the five forces.

If OF , FG , GH , HI , and IK represent the distances and directions that the forces would move the body, if acting separately, OK is the direction and distance of movement of the body when all the forces act together.

880. From what has been said before, it is seen that any number of forces acting on a body at the same point, or having their lines of action pass through the same point, can be replaced by a *single force* (resultant), whose line of action shall pass through that point.

881. Heretofore, it has been assumed that the forces acted upon a single point on the *surface* of the body, but it will make no difference where they act, so long as the lines of action of all the forces intersect at a *single point*, either within or without the body, only so that the resultant can be drawn through the *point of intersection*. If two forces act upon a body in the same straight line and in the same direction, their *resultant* is the *sum of the two forces*; but, if they act in opposite directions, their *resultant* is the *difference of the two forces*, and its direction is the same as that of the greater force. If they are equal and opposite, the *resultant* is *zero*, or one force just balances the other.

EXAMPLE.—Find the resultant of the forces whose lines of action pass through a single point, as shown in Fig. 117.

SOLUTION.—Take any convenient point g , and draw a line gf , parallel to one of the forces, say the one marked 40, making it equal in length to 40 pounds on the scale, and indicate its direction by the arrow-head. Take some other force—the one marked 37 will be convenient; the line fe represents this force. From the point e , draw a line parallel to some other force; say the one marked 29, and make it equal in magnitude and direction to it. So continue with the other forces, taking care that the general direction around the polygon is not

changed. The last force drawn in the figure is ab , representing the force marked 25. Join the points a and g ; then, ag is the resultant of all the forces shown in the figure. Its direction is from g to a op-

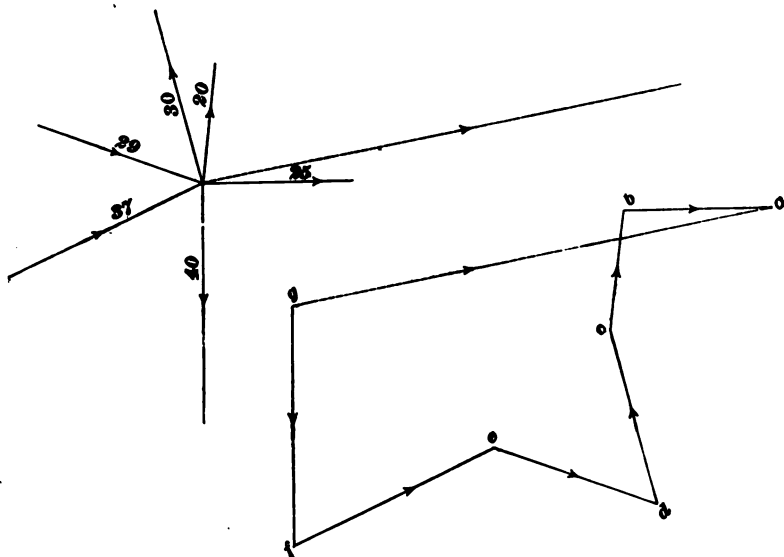


FIG. 117.

posed to the general direction of the others around the polygon. It does not matter in what order the different forces are taken, the resultant will be the same in magnitude and direction, if the work is done correctly.

882. The various methods of finding the resultant of several forces are all grouped under one head : The **composition of forces**.

THE RESOLUTION OF FORCES.

883. Since two forces can be combined to form a single resultant force, we may also treat a single force as if it were the resultant of two forces, whose action upon a body will be the same as that of a single force. Thus, in Fig. 118, the force OA may be resolved into two forces, OB and BA , whose directions are opposed to OA .

If the force OA acts upon a body, moving or at rest upon a horizontal plane, and the resolved force OB is vertical, and BA horizontal, OB , measured to the same scale as

OA , is the magnitude of that part of OA which pushes the body *downwards*, while BA is the magnitude of that part

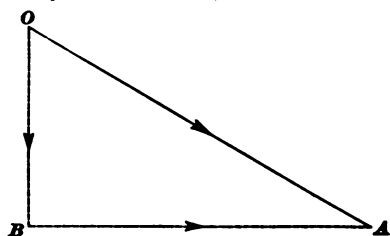


FIG. 118.

of the force OA which is exerted in pushing the body in a *horizontal* direction. OB and BA are called the **components** of the force OA , and when these components are vertical and horizontal, as in the present case, they are called the *vertical component* and the *horizontal component* of the force OA .

884. It frequently happens that the position, magnitude and direction of a certain force is known, and that it is desired to know the effect of the force in some direction, other than that in which it acts. Thus, in Fig. 118, suppose that OA represents, to some scale, the magnitude, direction, and line of action of a force acting upon a body at A , and that it is desired to know what effect OA produces in the direction BA . Now BA , instead of being horizontal, as in the cut, may have any direction. To find the value of the component of OA which acts in the direction BA , we employ the following rule:

Rule V.—*From one extremity of the line representing the given force, draw a line parallel to the direction in which it is desired that the component shall act; from the other extremity of the given force, draw a line perpendicular to the component first drawn, and intersecting it. The length of the component, measured from the point of intersection to the intersection of the component with the given force, will be magnitude of the effect produced by the given force in the required direction.*

Thus, suppose OA , Fig. 118, represents a force acting upon a body resting upon a horizontal plane, and it is desired to know what *vertical pressure* OA produces on the body. Here the desired direction is vertical; hence, from one extremity, as O , draw OB parallel to the desired direction (vertical in this case), and from the other extremity,

draw AB perpendicular to OB , and intersecting OB at B . Then OB , when measured to the same scale as OA , will be the value of the vertical pressure produced by OA .

EXAMPLE.—If a body weighing 200 pounds rests upon an inclined plane whose angle of inclination to the horizontal is 18° , what force does it exert perpendicular to the plane, and what force does it exert parallel to the plane, tending to slide downwards?

SOLUTION.—Let ABC , Fig. 119, be the plane, the angle A being equal to 18° , and let W be the weight. Draw a vertical line $FD = 200$ pounds, to represent the magnitude of the weight. Through F , draw FE parallel to AB , and through D draw DE perpendicular to EF , the two lines intersecting at E . FD is now resolved into two components, one, FE , tending to pull the weight downwards, and the other, ED , acting as a perpendicular pressure on the plane.

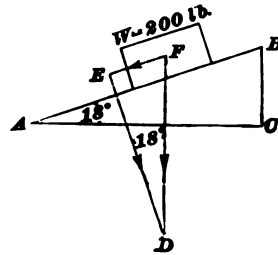


FIG. 119.

Since FD is perpendicular to AC , and ED is perpendicular to AB , the angle $D = \text{angle } A = 18^\circ$.

Hence, $FE = 200 \times \sin 18^\circ = 200 \times .30902 = 61.804$ pounds, and $ED = 200 \times \cos 18^\circ = 200 \times .95106 = 190.212$ pounds.

Force parallel to the plane = 61.804 pounds.

Force perpendicular to the plane = 190.212 pounds. } Ans.

DYNAMICS.

885. Dynamics may be defined as that branch of Mechanics which deals with bodies moving with a variable velocity. In Elementary Mechanics we shall consider only falling bodies and centrifugal force.

GRAVITATION.

886. Every body in the universe exerts a certain attractive force on every other body, which tends to draw the two bodies together. This attractive force is called **gravitation**.

If a body is held in the hand, a downward pull is felt, and if let go of, it will fall to the ground. This pull is commonly called *weight*, but it really is the attraction between the earth and the body.

887. Force of gravity is a term used to denote the attraction between the earth and bodies upon or near its surface. It always acts in a straight line between the center of the body and the center of the earth. The force of gravity varies at points on the earth's surface.

It is slightly less on the top of a high mountain than at the level of the sea. For this reason, the weight of a body also varies. But if the weight of a body at any place be divided by the force of gravity at that place, the result is called the *mass* of the body.

888. The mass of a body is the measure of the actual amount of matter that it contains, and is *always the same*.

If the mass of the body be represented by m , its weight by W , and the force of gravity at the place where the body was weighed, by g , we have

$$\text{mass} = \frac{\text{weight of body}}{\text{force of gravity}}, \text{ or } m = \frac{W}{g}. \quad (10.)$$

889. Law of Gravitation:—

The force of attraction by which one body tends to draw another body towards it, is directly proportional to its mass, and inversely proportional to the square of the distance between their centers.

890. Laws of Weight:—

Bodies weigh most at the surface of the earth. Below the surface, the weight decreases as the distance to the center decreases.

Above the surface the weight decreases as the square of the distance increases.

ILLUSTRATION.—If the earth's radius is 4,000 miles, a body that weighs 100 pounds at the surface will weigh nothing at the center, since it is attracted in every direction with equal force. At 1,000 miles from the center, it will weigh 25 pounds, since

$$4,000 : 1,000 = 100 : 25.$$

At 2,000 miles from the center, it will weigh 50 pounds, since

$$4,000 : 2,000 = 100 : 50.$$

At 3,000 miles from the center, it will weigh 75 pounds, and at the surface, or 4,000 miles from the center, it will weigh 100 pounds. If carried still higher, say 1,000 miles from the surface, or 5,000 miles from the center of the earth, it will weigh 64 pounds, since

$$5,000^2 : 4,000^2 = 100 : 64.$$

At 4,000 miles from the surface, it will weigh 25 pounds, since

$$8,000^2 : 4,000^2 = 100 : 25.$$

891. Formulas for Gravity Problems:—

Let W = weight of body at the surface;

w = weight of a body at a given distance above or below the surface;

d = distance between the center of the earth and the center of the body;

R = radius of the earth = 4,000 miles.

Formula for weight when the body is below the surface:

$$w R = d W. \quad (11.)$$

Formula for weight when the body is above the surface:

$$w d^2 = W R^2. \quad (12.)$$

EXAMPLE.—How far below the surface of the earth will a 25-pound ball weigh 9 pounds?

SOLUTION.—Use formula 11, $w R = d W$.

Substituting the values of R , W , and w , we have

$$9 \times 4,000 = d \times 25, \text{ or } d = \frac{9 \times 4,000}{25} = 1,440 \text{ miles from the center. Ans.}$$

EXAMPLE.—If a body weighs 700 pounds at the surface of the earth, at what distance above the earth's surface will it weigh 112 pounds?

SOLUTION.—Use formula 12, $w d^2 = W R^2$.

Substituting the values of R , W , and w , we have

$$112 \times d^2 = 700 \times 4,000^2, \text{ or } d = \sqrt{\frac{700 \times 4,000^2}{112}} = 10,000 \text{ miles.}$$

Therefore, $10,000 - 4,000 = 6,000$ miles above the earth's surface. Ans.

EXAMPLE.—The top of Mt. Hercules was said to be 32,000 feet, say 6 miles above the level of the sea. If a body weighs 1,000 pounds at sea-level, what would it weigh if carried to the top of the mountain?

SOLUTION.— $w d^2 = W R^2$, or $w \times 4,006^2 = 1,000 \times 4,000^2$; whence,

$$w = \frac{4,000^2 \times 1,000}{4,006^2} = 997 \text{ pounds. Ans.}$$

EXAMPLES FOR PRACTICE.

1. How much would 1,000 tons of coal weigh one mile below the surface? Ans. 1,999,500 lb.
2. How much would the coal in example 1 weigh one mile above the surface? Ans. 1,999,000 lb., nearly.
3. How far above the earth's surface would it be necessary to carry a body in order that it may weigh only half as much? Ans. 1,656.854 miles, nearly.
4. A man weighs 160 pounds at the surface; how much will he weigh 50 miles below the surface? Ans. 158 lb.
5. If a body weighs 100 pounds 400 miles above the earth's surface, how much will it weigh at the surface? Ans. 121 lb.

NOTE.—Use 4,000 miles as the radius of the earth.

FALLING BODIES.

892. If a leaden ball and a piece of paper are dropped from the same height, the ball would strike the ground first.



FIG. 120.

This is not because the leaden ball is the heavier, but because the resistance of the air has a greater retarding effect upon the paper than upon the ball. If we place this same leaden ball and a piece of paper in a glass tube, Fig. 120, from which all of the air has been exhausted, it would be found that when the tube was inverted, both would drop to the bottom in exactly the same time. This experiment proves that it was only the resistance of the air that caused the ball to reach the ground first, in the former experiment. This resistance of the air may be nearly equalized by making the two bodies of the same shape and size. For example, if a wooden and an iron ball, having equal diameters, were dropped from the same height, they would strike the ground at almost

exactly the same instant, although the iron ball might be ten times as heavy as the wooden ball.

893. Suppose there were several leaden balls, as shown in Fig. 121, at a ; it is obvious that if they were dropped

together, all would strike the ground at the same time. If the balls were melted together into one ball, as *b*, they would still fall together, and strike the ground in the same time as before.



FIG. 121.

Since a number of horses cannot run a mile in less time than a single horse, so 100 pounds can fall no further in a given time than one pound can.

894. Acceleration is the rate of change of velocity. If a force acts upon a body free to move, then, according to the first law of motion, it will move forever with the same velocity unless acted upon by another force.

Suppose that, at the end of one second, the same force were to act again, the velocity at the end of the second second would be twice as great as at the end of the first second. If the same force were to act again, the velocity at the end of the third second would be three times that at the end of the first second. So, if a constant force acts upon a body free to move, the velocity of the body at the end of any time will be the velocity at the end of one second, multiplied by the number of seconds.

895. This constant force is called a **constant accelerating force**, or **constant retarding force**, according as the velocity is constantly *increased* or *decreased*.

If a body is dropped from a high tower, the velocity with which it approaches the ground will be constantly increased or accelerated; for the attraction of the earth, or force of gravity, is constant, and acts upon the body as a constant accelerating force. It has been found by careful experiments that this force of gravity, or constant accelerating force, on a freely falling body, is equivalent to giving the body a velocity of 32.16 feet in one second; it is always denoted by *g*. As was mentioned before, *g* varies at different points of the earth, being 32.0902 at the equator, and 32.2549 at the poles. Its value for this latitude (about 41° 25' north) is very nearly 32.16, and this value should always be used in solving problems. It has also been found by

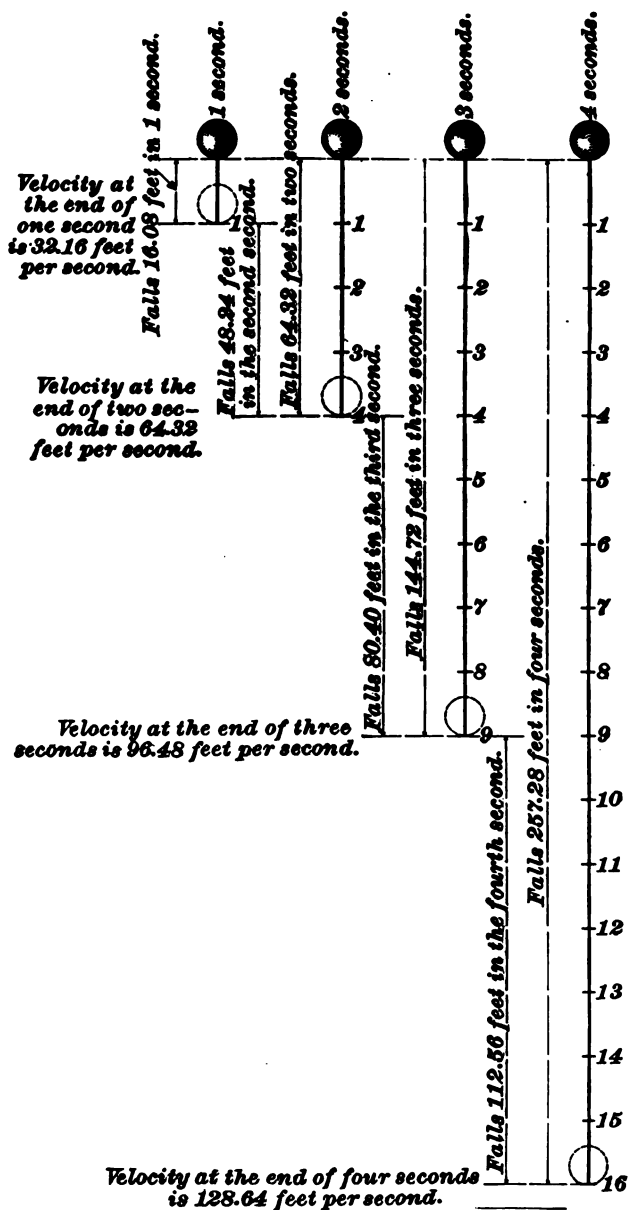


FIG. 122.

experiment that a freely falling body starting from rest will have fallen 16.08 feet at the end of the first second; 64.32 feet at the end of the second second; 144.72 feet at the end of the third second; 257.28 feet at the end of the fourth second, etc., all of which are shown in the diagram, Fig. 122.

Since $\frac{64.32}{16.08} = 4 = 2^2$; $\frac{144.72}{16.08} = 9 = 3^2$; $\frac{257.28}{16.08} = 16 = 4^2$, and $2^2, 3^2, 4^2$ are the squares of the number of seconds during which the body falls, it is easy to see that the space through which a body free to move will fall in a given time is equal to 16.08 multiplied by the square of the time in seconds.

Since $16.08 = \frac{32.16}{2} = \frac{1}{2}g$, the space $= \frac{1}{2}g \times$ square of time in seconds.

896. Formulas for Falling Bodies :—

Let g = force of gravity = constant accelerating force due to the attraction of the earth;

t = number of seconds the body falls;

v = velocity at the end of the time t ;

h = distance that a body falls during the time t .

$$v = gt. \quad (13.)$$

That is, the velocity acquired by a freely falling body at the end of t seconds equals 32.16, multiplied by the time in seconds.

EXAMPLE.—What is the velocity of a body after it has fallen four seconds, assuming that the air offered no resistance?

SOLUTION.—Using formula 13,

$$v = gt = 32.16 \times 4 = 128.64 \text{ feet per second. Ans.}$$

$$t = \frac{v}{g}. \quad (14.)$$

That is, the number of seconds during which a body must have fallen to acquire a given velocity equals the given velocity in feet per second, divided by 32.16.

EXAMPLE.—A falling body has a velocity of 192.96 feet per second; how long had it been falling at that instant?

SOLUTION.—Using formula 14,

$$t = \frac{v}{g} = \frac{192.96}{32.16} = 6 \text{ seconds. Ans.}$$

$$h = \frac{v^2}{2g}. \quad (15.)$$

That is, the height from which a body must fall to acquire a given velocity equals the square of the given velocity, divided by 2×32.16 .

EXAMPLE.—From what height must a stone be dropped to acquire a velocity of 24,000 feet per minute?

SOLUTION.— $24,000 \div 60 = 400$ feet per second. Using formula 15,

$$h = \frac{v^2}{2g} = \frac{400^2}{2 \times 32.16} = \frac{160,000}{64.32} = 2,487.56 \text{ feet. Ans.}$$

$$v = \sqrt{2gh}. \quad (16.)$$

That is, the velocity that a body will acquire in falling through a given height equals the square root of the product of twice 32.16, and the given height.

EXAMPLE.—A body falls from a height of 400 feet; what will be its velocity at the end of its fall?

SOLUTION.—Using formula 16,

$$v = \sqrt{2gh} = \sqrt{2 \times 32.16 \times 400} = 160.4 \text{ feet per second. Ans.}$$

$$h = \frac{1}{2}gt^2. \quad (17.)$$

That is, the distance a body will fall in a given time equals $32.16 \div 2$, multiplied by the square of the number of seconds.

EXAMPLE.—How far will a body fall in 10 seconds?

SOLUTION.—Using formula 17,

$$h = \frac{1}{2}gt^2 = \frac{1}{2} \times 32.16 \times 10^2 = 1,608 \text{ feet. Ans.}$$

$$t = \sqrt{\frac{2h}{g}}. \quad (18.)$$

That is, the time it will take a body to fall through a given height equals the square root of twice the height divided by 32.16.

EXAMPLE.—How long will it take a body to fall 4,116.48 feet?

SOLUTION.—Using formula 18,

$$t = \sqrt{\frac{2 \times 4,116.48}{32.16}} = 16 \text{ seconds. Ans.}$$

897. A body thrown vertically upwards starts with a certain velocity called the **initial velocity**. In this case gravity acts as a constant retarding force. The formulas given above will also apply in this case.

EXAMPLE.—If a cannon ball is shot vertically upwards with an initial velocity of 2,000 feet per second, (a) how high will it go? (b) How long a time must elapse before it reaches the earth again?

SOLUTION.—(a) Using formula 15,

$$h = \frac{v^2}{2g} = \frac{2,000^2}{2 \times 32.16} = 62,189 \text{ feet, nearly,} = 11.778 \text{ miles. Ans.}$$

To find the time it takes to reach a height of 62,189 feet, use formula 14.

$$t = \frac{v}{g} = \frac{2,000}{32.16} = 62.19 \text{ seconds.}$$

Since it will take the same length of time to fall to the ground, the total time will be $62.19 \times 2 = 124.38$ seconds = 2 minutes 4.38 seconds.
Ans.

PROJECTILES.

898. Any body thrown into the air is a **projectile**, and is acted upon by three forces—the original or initial force, the force of gravity, and the resistance of the air. We shall here consider only those projectiles which are thrown horizontally.

899. The **range** is the horizontal distance between the starting point and the point where the body strikes the ground. In Fig. 123, suppose that *A* represents the starting point of the projectile, and that it is shot horizontally outwards in the direction of the arrow with a velocity of 70 feet per second. Now, if the resistance of the air be neglected, the velocity in the horizontal direction will be uniform, and the projectile will pass over equal spaces in equal times. Let *A 1* represent 70 feet, or the space passed over in one second. At the end of five seconds, if gravity had not acted upon the projectile, it

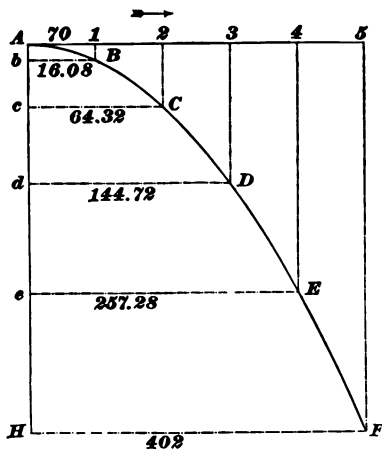


FIG. 123.

would have been at 5, but as gravity has acted, it *falls* 16.08 feet the first second; at the end of the second second it has fallen 64.32 feet, etc.

Let $A b$ represent the fall in one second—that is, 16.08 feet, drawn to the same scale as $A I$, which represents 70 feet. Now, complete the parallelogram $A I B b$, and B will be the point which the projectile has reached at the end of one second. If $A c$ represents 64.32 feet, and the parallelogram $A c C c$ is completed, the projectile will be at C at the end of the second second. Proceeding in this manner, find the points D, E , and F , the positions of the projectile at the end of 3, 4, and 5 seconds, respectively. Drawing the curve $A B C D E F$ through the points thus found, it represents the path of the projectile. This curve is called a **parabola**.

The distance $H F$ is the *range*, and, as is easily seen, *equals the time in seconds multiplied by the original velocity in feet per second*.

900. If the height $A H$ and the initial velocity are given, and it is desired to find the range $H F$, *calculate the time that it will take to fall through a height equal to the given height, and multiply the time thus found by the initial velocity*.

EXAMPLE.—A cannon ball is fired in a horizontal direction with an initial velocity of 1,500 feet per second. If the mouth of the cannon is 25 feet above the ground, what is its range?

SOLUTION.—Applying formula **18**, Art. **896**,

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 25}{32.16}} = 1.247 \text{ seconds, nearly.}$$

$$\text{Range} = v t = 1,500 \times 1.247 = 1,870.5 \text{ feet. Ans.}$$

EXAMPLE.—A projectile has an initial velocity of 90 feet per second. If it is desired to strike an object 15 feet away, how far below the horizontal line of direction must the object be located?

SOLUTION.—The object must be located as far below as the distance that the body would fall, through the action of gravity, during the time it would take in passing over a distance of 15 feet at a velocity of 90 feet per second.

Hence, $15 \div 90 = \frac{1}{6}$ of a second. Applying formula **17**, Art. **896**, $h = \frac{1}{2} g t^2 = \frac{1}{2} \times 32.16 \times (\frac{1}{6})^2 = .447$ foot, nearly, = 5.36 inches. Ans.

EXAMPLES FOR PRACTICE.

1. A body starts from a state of rest, and falls freely for nine seconds; how far will it fall? Ans. 1,802.48 ft.
2. What velocity must a body have in order to carry it upwards 500 feet, vertically? Ans. 179.33 ft. per sec.
3. A baseball is thrown vertically upwards to a height of 200 feet; how long a time must elapse before it strikes the ground? Ans. 7.05 sec.
4. What will be the velocity of a freely falling body at the end of 6 seconds? Ans. 192.96 ft. per sec.
5. A baseball is thrown horizontally 5 feet above the ground, with a velocity of 80 feet per second; what is its range? Ans. 44.61 ft.
6. A leaden bullet falls from a tower 100 feet high; with what velocity will it strike the ground? Ans. 80.2 ft. per sec.
7. A bullet is dropped from a high tower. If it takes $4\frac{1}{2}$ seconds to reach the ground, how high is the tower? Ans. 290.445 ft.
8. A freely falling body has a velocity of 400 feet per second; how long has it been falling? Ans. 12.438 sec.

CENTRIFUGAL FORCE.

901. If a body be fastened to a string and whirled so as to give it a circular motion, there will be a pull on the string, which will be greater or less according as the velocity increases or decreases. The cause of this pull on the string will now be explained.

Suppose that the body is revolved horizontally, so that the action of gravity upon it will always be the same. According to the first law of motion, a body put in motion tends to move in a straight line unless acted upon by some other force, causing a change in the direction. When a body moves in a circle the force that causes it to move in a circle instead of a straight line is exactly equal to the tension of the string. If the string were cut, the pulling force that drew it away from the straight line would be removed and the body would then "fly off at a tangent"—that is, it would move in a straight line tangent to the circle, as shown in Fig. 124.

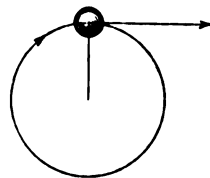


FIG. 124.

902. Since, according to the third law of motion, every action has an equal and opposite reaction, we call that force

which acts as an equal and opposite force to the pull of the string the **centrifugal force**, and it acts *away* from the center of motion.

903. The other force or tension of the string is called the **centripetal force**, and it acts *towards* the center of motion. It is evident that these two forces acting in opposite directions tend to pull the string apart, and, if the velocity be increased sufficiently, the string will break. It is also evident that no body can revolve without generating centrifugal force. The value of the centrifugal force of any revolving body, expressed in pounds, is

$$F = .00034 W R N^2, \quad (19.)$$

in which F = centrifugal force;

W = total weight of body in pounds;

R = radius, usually taken as the distance between the center of motion and the center of gravity of the revolving body, in feet;

N = number of revolutions per *minut*c.

904. In calculating the centrifugal force tending to burst a fly-wheel, it is the usual practice to consider one-half the rim of the wheel only, and not to take the arms and hub of the wheel into account. In this case, R is taken as the *distance between the inside edge of the rim and the center of the shaft*.

NOTE.—The general formula for centrifugal force is $F = \frac{m v^2}{R}$, where m = the mass of the revolving body, v = velocity of center of gravity of body in feet per second, and R = radius, as above. Formula 19, Art. 903, is easily derived from this. Thus: $m = \frac{W}{g}$; $N = \frac{60 v}{2\pi R}$, or $v = \frac{2\pi R N}{60}$; hence, $F = \frac{m}{R} v^2 = \frac{W}{g R} \left(\frac{2\pi R N}{60} \right)^2 = .00034 W R N^2$.

EXAMPLE.—What would be the centrifugal force tending to burst a cast-iron fly-wheel whose outside diameter was 10 feet, width of face 20 inches, and thickness of rim 6 inches, turning at the rate of 80 revolutions per minute?

SOLUTION.—First calculate the weight of one-half the rim. The diameter of the rim = $10 \times 12 = 120$ inches; the diameter of the circle midway between the inside and outside diameters of the rim =

$120 - 6 = 114$ inches. The number of cubic inches in the rim $= 114 \times 3.1416 \times 20 \times 6 = 42,977$ cubic inches. $42,977 \times .261 \times \frac{1}{2} = 5,608.5$ pounds $=$ weight $= W$. $R = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ feet. $N = 80$.

Hence, $F = .00084 WRN^2 = .00084 \times 5,608.5 \times \frac{1}{4} \times 80^2 = 54,918.4$ pounds. Ans.

STATICS.

905. Statics may be defined as that branch of Mechanics which treats of bodies at rest or of bodies moving with a *uniform* velocity, when these bodies are acted upon by forces. A body is in **static equilibrium** when the resultant of *all* of the forces acting upon the body is zero.

MOMENTS OF FORCES.

906. If from any point O , Fig. 125, a perpendicular be drawn to the line of action of a force, the product of the magnitude of the force and the length of the perpendicular is called the **moment of the force about the point O** .

Thus, in the figure, the moment of the force F' about the point O is $F' \times OB$; of the force F'' about the point O is $F'' \times OA$, and of F''' is $F''' \times OC$.

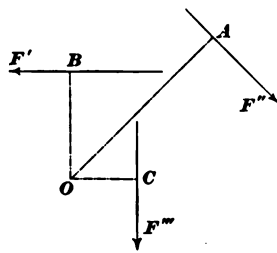


FIG. 125.

907. The use of the moment will be explained further on, when the necessity arises for using it. The point O is called the **center of moments**.

908. When two *equal* forces act in parallel lines, but in opposite directions, they constitute what is called a **couple**.

909. In Fig. 126, the equal and parallel forces F' and F'' , acting in opposite directions (one up and the other down), form a couple. It is easy to see that if they were joined by a connection, as AB , that they would tend to turn AB about the point C , midway between F' and F'' . The moment

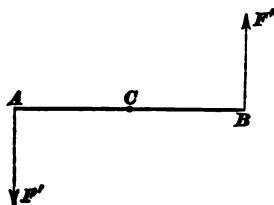


FIG. 126.

of a couple about *any* point is always the same, and is equal to the *product of one of the equal forces into the perpendicular distance between the two forces.*

Thus, the moment of the couple in the figure equals F' or F'' multiplied by AB . An example of a couple would be a wrench applied to a nut. Here, two opposite and parallel sides of the wrench act in parallel, but opposite, directions, against two parallel sides of the nut.

CENTER OF GRAVITY.

910. *The center of gravity of a body is that point at which the body may be balanced, or it is the point at which the whole weight of a body may be considered as concentrated.*

In a moving body, the line described by its center of gravity is always taken as the path of the body. In finding the distance that a body has moved, the distance that the center of gravity has moved is taken.

911. The definition of the center of gravity of a body may be applied to a system of bodies, if they are considered as being connected at their centers of gravity.

If w and W , Fig. 127, be two bodies of known weights,

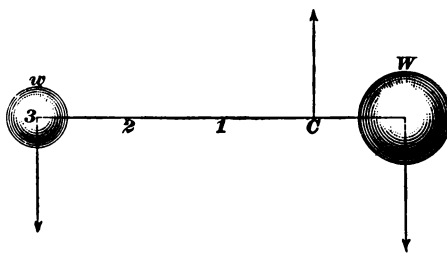


FIG. 127.

readily determined, as follows: Take C as the center of moments; then, since the weights are to balance each other, the moment of W about C must equal the moment of w about C ; or, in other words,
 $W \times CW = w \times Cw$.

If the distance between the centers of gravity of W and w is known, it is very easy to find Cw and CW . For

Let l = the distance wW between the centers of the bodies ;

l_1 = the short arm CW ;

w = weight of small body ;

W = weight of large body.

Then, since $wC = l - l_1$, we have, taking the moments about the point C ,

$$Wl_1 = w(l - l_1) = wl - wl_1 ; \text{ whence,}$$

$$Wl_1 + wl_1 = (W + w)l_1 = wl, \text{ or}$$

$$l_1 = \frac{wl}{W + w}. \quad (20.)$$

EXAMPLE.—In Fig. 127, $w = 10$ pounds, $W = 30$ pounds, and the distance between their centers of gravity is 36 inches; where is the center of gravity of both bodies situated ?

SOLUTION.—Applying formula 20,

$$CW = l_1 = \frac{wl}{W + w} = \frac{10 \times 36}{30 + 10} = 9 \text{ inches;}$$

hence, the center of gravity is 9 inches from the center of the larger body. **Ans.**

The general method for finding the short arm CW is, then, as follows : *Multiply the weight of the smaller body by the distance between the centers of the two bodies, and divide this product by the sum of the weights of the two bodies.*

912. It is now very easy to extend this principle, to the finding of the center of gravity of any number of bodies when their weights and the distances apart of their centers of gravity are known, by applying the principle of finding the resultant of several forces; that is, by finding the center of gravity of two of the bodies, as W_1 and W_4 in Fig. 128, at C_1 .

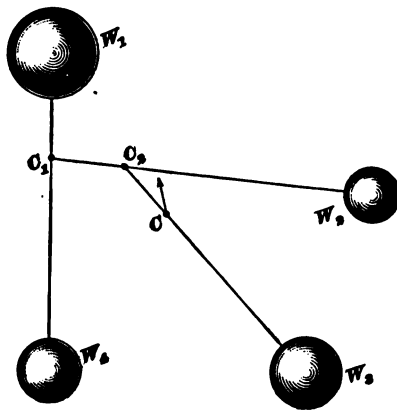


FIG. 128.

Assume that the weight of both bodies is concentrated at C_1 , and find the center of gravity of this combined weight at C_1 and of W_2 to be at C_2 ; then, find that the center of gravity of the combined weights of W_1 , W_2 , and W_3 (concentrated at C_2) and W_4 to be at C , and C will be the center of gravity of the four bodies.

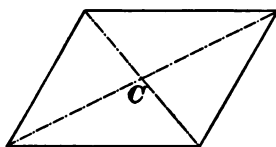


FIG. 129.

913. To find the center of gravity of any parallelogram: *Draw the two diagonals, Fig. 129, and their point of intersection C will be the center of gravity.*

914. To find the center of gravity of a triangle, as ABC , Fig. 130: *From any vertex, as A , draw a line to the middle point D of the opposite side BC . From one of the other vertices, as C , draw a line to F , the middle point of the opposite side AB ; the point of intersection, O , of these two lines, is the center of gravity.*

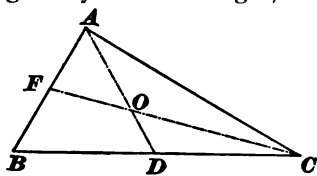


FIG. 130.

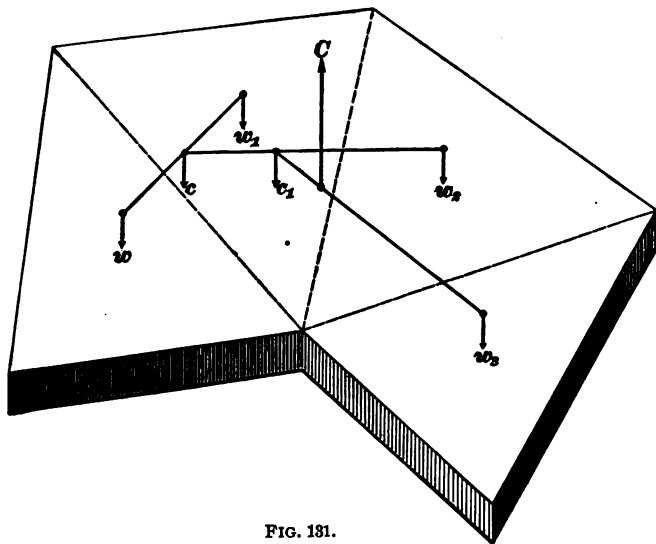


FIG. 131.

It is also true that the distance $DO = \frac{1}{3} DA$, and that

$FO = \frac{1}{3} FC$, and the center of gravity could have been found by drawing from any vertex a line to the middle point of the opposite side, and measuring back from that side $\frac{1}{3}$ of the length of the line.

The center of gravity of any regular plane figure is the same as the geometrical center.

915. To find the center of gravity of any irregular plane figure, but of uniform thickness throughout, divide one of the parallel surfaces into triangles, parallelograms, circles, ellipses, etc., according to the shape of the figure; find the area and center of gravity of each part separately, and combine the centers of gravity thus found, as in the case of more than two bodies whose weights were known, except that the area of each part is used instead of their weights. See Fig. 131.

916. Center of Gravity of a Solid.—In a body free to move, the center of gravity will lie in a vertical plumb

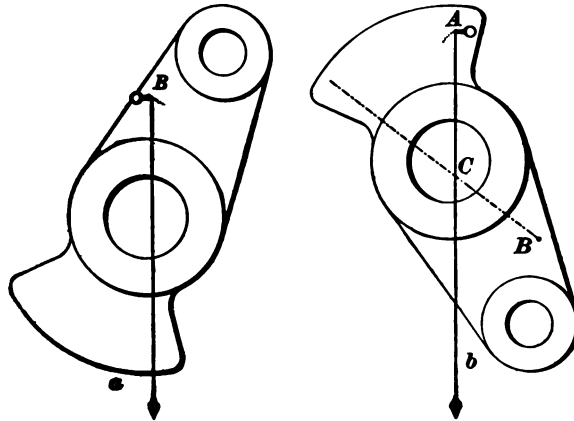


FIG. 132.

line drawn through the point of support. Therefore, to find the position of the center of gravity of an irregular solid, as the crank, Fig. 132, suspend it at some point, as B , so that it will move freely. Drop a plumb line from the point of suspension, and mark its direction. Suspend the body at another point, as A , and repeat the process. The

intersection of the two lines will be directly over the center of gravity.

Since the center of gravity depends wholly upon the shape and weight of a body, it may be without the body, as in the case of a circular ring, whose center of gravity is at the center of the circumference of the ring.

EQUILIBRIUM.

917. When a body is at rest, all of the forces which act upon it are said to *balance* one another, or to be in **equilibrium**. The most important of the forces is gravity, which acts upon every molecule of the body.

918. There are three states of equilibrium : **Stable, unstable, and neutral.**

919. A body is in **stable equilibrium** when, if slightly displaced from its position of rest, *it tends to return to that position.*

For example, a cube, a cone resting on its base, a pendulum, etc.

If a body is in **stable equilibrium**, *its center of gravity is raised when it is displaced.*

920. A body is in **unstable equilibrium** when, if slightly displaced from its position of rest, *it tends to fall farther from that position.*

For example, a cone standing upon its point, an egg balanced upon its end, etc.

Any movement, however slight, lowers *the center of gravity* when the body is in unstable equilibrium.

921. A body is in **neutral equilibrium** when it has *no tendency to move either way*, in the direction of its motion, after being slightly displaced.

For example, a sphere of uniform density ; a cone resting on its side.

922. A vertical line drawn through the center of gravity of a body is called the **line of direction**. So long as the line of direction falls within the base, the body will stand. When the line of direction falls without the base, the body will fall.

Let $A C B$, Fig. 133, be a cylinder whose base is oblique to the center line $B O D$; and let O be the center of gravity of this cylinder.

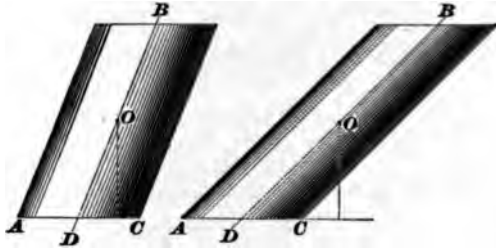


FIG. 133.

So long as the perpendicular through O falls between A and C , the cylinder will stand, but the instant that it falls without the base, the cylinder will fall.

The center of gravity of a body always tends to seek its lowest point.

EXAMPLES FOR PRACTICE.

1. There are three weights in a straight line. The first weighs 40 lb.; the second, 16 lb., and the third, 50 lb. Distance between the first and second is 6 ft., and between the second and third, 10 ft. Where is the center of gravity? Ans. 8 ft. 5.434 in. from the 40-lb. weight.
2. Find the perpendicular distance between the center of gravity and the longer side of a triangle whose sides are 7 ft., 10 ft., and 15 ft. long. Solve graphically. Ans. 1.31 ft.
3. A rectangle, 2 ft. long and 1 ft. wide, has equal weights of 50 lb. each, suspended from two of its diagonally opposite corners. A weight of 60 lb. and another of 80 lb. are suspended from the other two corners. Supposing the rectangle to be without weight, where is the center of gravity?
Ans. { On the diagonal joining the 60-lb. and 80-lb. weights, 1.118 in. from the center.
4. Find the center of gravity of a quadrilateral whose sides are 14, 15, 16, and 18 in. long, the angle between the 18 and 16 in. sides being 45° . Give the perpendicular distance from the 18 in. side. Ans. 5.46 in.

SIMPLE MACHINES.

THE LEVER.

923. A **lever** is a bar capable of being turned about a pin, pivot or point, as in Figs. 134, 135 and 136.

924. The object W to be lifted is called the **weight**; the force used P is called the **power**; and the point or pivot F is called the **fulcrum**.

925. That part of the lever between the weight and the fulcrum, or Fb , is called the **weight arm**, and the part between the power and the fulcrum, or Fc , is called the **power arm**.

926. Take the fulcrum, or point F , as the center of moments; then, in order that the lever shall be in equilibrium, the moment of P about F , or $P \times Fc$, must equal the moment of W about F , or $W \times Fb$. That is, $P \times Fc = W \times Fb$, or, in other words, *the power multiplied by the power arm equals the weight multiplied by the weight arm*.

927. If F be taken as the center of a circle, and arcs be described through b and c , it will be seen that, if the weight arm is moved through a certain angle, the power arm will move through the same angle; also, that the vertical distance that W moves will be the sine of this angle, in a circle whose radius is the weight arm, and that the vertical distance that P moves will be the sine of the same angle in a circle whose radius is the power arm. From this it is seen that the power arm is proportional to the distance through

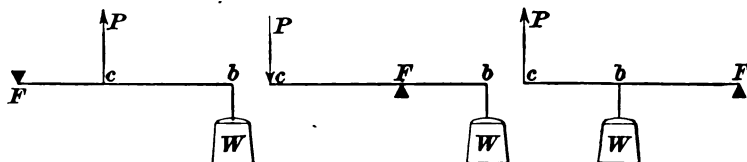


FIG. 134.

FIG. 135.

FIG. 136.

which the power moves, and the weight arm is proportional to the distance through which the weight moves.

Hence, instead of writing $P \times Fc = W \times Fb$, we might have written it $P \times \text{distance through which } P \text{ moves} = W \times \text{distance through which } W \text{ moves}$. This is the general law of all machines, and can be applied to any mechanism, from the simple lever up to the most complicated arrangement. Stated in the form of a rule, it is as follows:

Rule VI.—*The power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.*

EXAMPLE.—If the weight arm of a lever is 6 inches long and the power arm is 4 feet long, how great a weight can be raised by a force of 20 pounds at the end of the power arm?

SOLUTION.—4 feet = 48 inches. Hence, $20 \times 48 = W \times 6$, or $W = 160$ pounds. Ans.

EXAMPLE.—(a) What is the ratio between the power and the weight in the last example? (b) In the last example, if P moves 24 inches, how far does W move? (c) What is the ratio between the two distances?

SOLUTION.—(a) $20 : 160 = 1 : 8$; that is, the weight moved is 8 times the power. Ans.

(b) $20 \times 24 = 160 \times x$. $x = \frac{480}{160} = 3$ inches, the distance that W moves. Ans.

(c) $3 : 24 = 1 : 8$, or the ratio is 1 : 8. Ans.

928. The law which governs the straight lever also governs the bent lever; but care must be taken to determine the true lengths of the lever arms, which are in every case the *perpendicular distances from the fulcrum to the line of direction of the weight or power*.

Thus, in Figs. 137, 138, 139, and 140, Fc in each case represents the power arm, and Fb the weight arm.

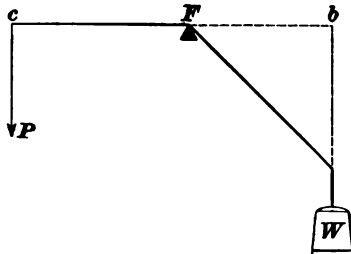


FIG. 137.

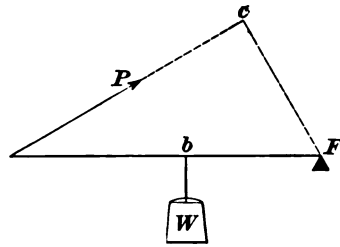


FIG. 138.

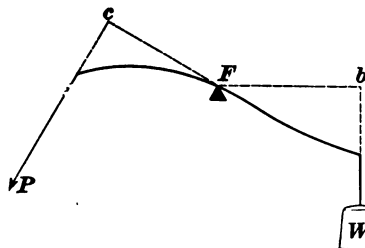


FIG. 139.

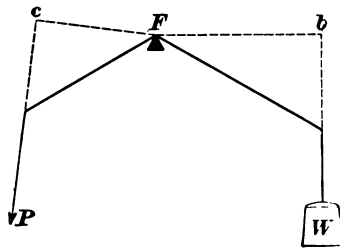


FIG. 140.

929. The Compound Lever.—A **compound lever** is a series of single levers arranged in such a manner that

when a force is applied to the first it is communicated to the second, and from this to the third, and so on.

Fig. 141 shows a compound lever. It will be seen that when a power is applied to the first lever at P it will be communicated to the second lever at P , from this to the third lever at P , and thus raise the weight W .

The weight which the power of the first lever could raise acts as the power of the second, and the weight which this could raise by means of the second lever acts as the power of the third lever, and so on, no matter how many single levers make up the compound lever.

In this case, as in every other, the power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.

Hence, if we move the P end of the lever, say, 4 inches, and the W end moves $\frac{1}{4}$ of an inch, we know that the ratio

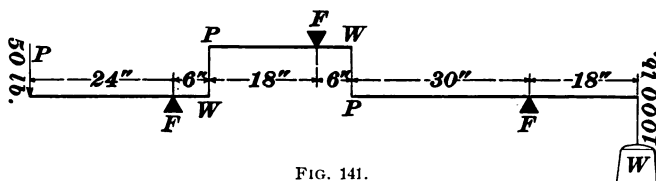


FIG. 141.

between P and W is the same as the ratio between $\frac{1}{4}$ and 4; that is, 1 to 20, and, hence, that 10 pounds at P would balance 200 pounds at W , without measuring the lengths of the different lever arms. If the lengths of the lever arms are known, the ratio between P and W may be readily obtained from the following rule:

RULE VII.—*The continued product of the power and each power arm equals the continued product of the weight and each weight arm.*

EXAMPLE.—If, in Fig. 141, the power arms $PF = 24$ inches, 18 inches, and 30 inches, and weight arms $WF = 6$ inches, 6 inches, and 18 inches, (a) how great a force must be applied at the free end P to raise 1,000 pounds at W ? (b) What is the ratio between P and W ?

SOLUTION.— $P \times 24 \times 18 \times 30 = 1,000 \times 6 \times 6 \times 18$,

$$\text{or } P = \frac{648,000}{12,960} = 50 \text{ pounds. Ans.}$$

$$50 : 1,000 = 1 : 20, \text{ or } P : W = 1 : 20. \text{ Ans.}$$

THE WHEEL AND AXLE.

930. The **wheel and axle** consists of *two cylinders of different diameters, rigidly connected*, so that they turn together about a common axis, as in Fig. 142. Then, as

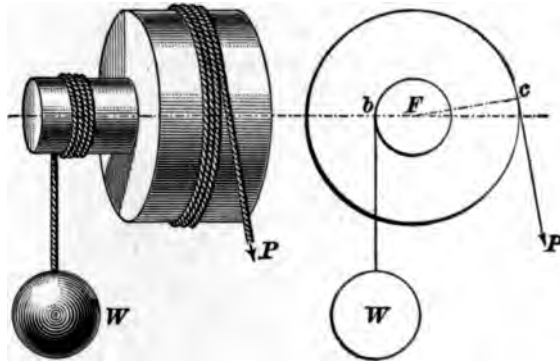


FIG. 142.

before, $P \times \text{distance through which it moves} = W \times \text{distance through which it moves}$; and, since these distances are proportional to the radii of the power cylinder and weight cylinder, $P \times Fc = W \times Fb$.

It is not necessary that an entire wheel be used; an arm, projection, radius, or anything which the power causes to revolve in a circle, may be considered as the wheel. Consequently, if it is desired to hoist a weight with a windlass, Fig. 143, the force is applied to the handle of the crank, and the distance between the center line of the crank-handle and the axis of the drum corresponds to the radius of the wheel.

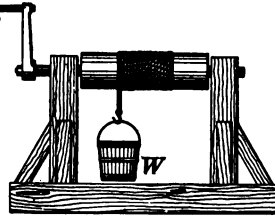


FIG. 143.

EXAMPLE.—If the distance between the center line of the handle and the axis of the drum, in Fig. 143, is 18 inches, and the diameter of the drum is 6 inches, what force will be required at P to raise a load of 300 pounds?

SOLUTION.— $P \times 18 = 300 \times \frac{6}{2}$, or $P = 50$. Ans.

931. Wheelwork.—A combination of wheels and axles, as in Fig. 144, is called a **train**. The wheel in a train to which motion is imparted from a wheel on another shaft, by such means as a belt or gearing, is called the **driven wheel** or **follower**; the wheel which imparts the motion is called the **driver**.

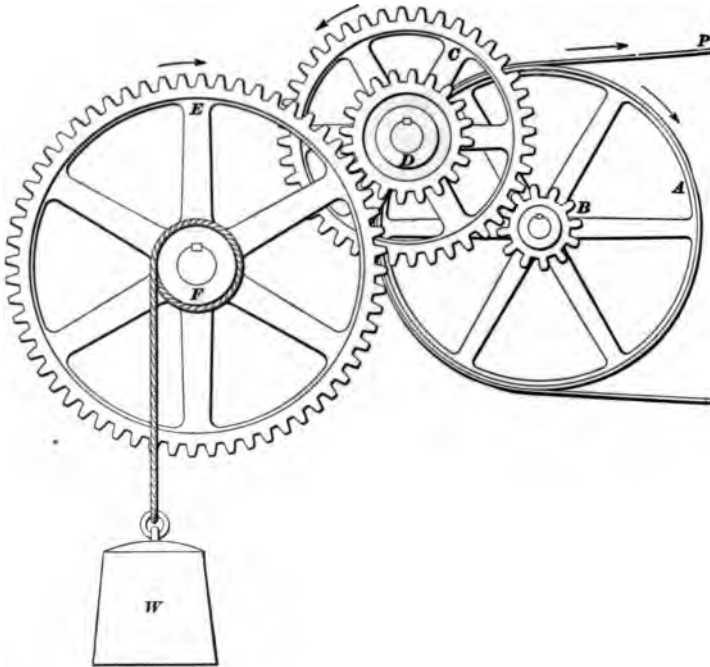


FIG. 144.

932. It will be seen that the wheel and axle bears the same relation to the train that the simple lever does to the compound lever; that is, *the continued product of the power and the radii of the driven wheels equals the continued product of the weight, the radius of the drum that moves the weight, and the radii of the drivers.*

EXAMPLE.—If the radius of the wheel *A* is 20 inches, of *C*, 15 inches, and of *E*, 24 inches; if the radius of the drum *F* is 4 inches, of the

pinion D , 5 inches, and of the pinion B , 4 inches, how great a weight will a force of 1 pound at P raise?

SOLUTION.— $1 \times 20 \times 15 \times 24 = W \times 4 \times 5 \times 4$, or $W = \frac{7,200}{80} = 90$ pounds. Ans.

933. Hence, also, if W were raised one inch, P would move 90 inches, or P would have to move 90 inches to raise W one inch. It is now clear that another great law has made itself manifest, and that is that, *whenever there is a gain in power without a corresponding increase in the initial force, there is a loss in speed.*

In the last example, if P were to move the entire 90 inches in one second, W would move only 1 inch in one second. The same principle may be applied to any machine.

THE PULLEY.

934. A **pulley** is a wheel turning on an axle, over which a cord, chain, or band is passed in order to transmit the force through the cord, chain, or band.

935. The frame which supports the axle of the pulley is called the **block**.

936. A **fixed pulley** is one whose block is not movable, as in Fig. 145. In this case, if the weight W be lifted by pulling down P , the other end of the cord W' will evidently move the same distance upwards that P moves downwards; hence, P must equal W .

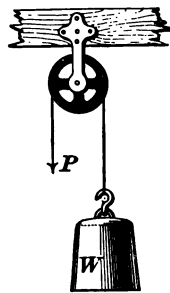


FIG. 145.

937. A **movable pulley** is one whose block is movable, as in Fig. 146. One end of the cord is fastened to the beam, and the weight is suspended from the pulley, the other end of the cord being drawn up by the application of a force P . A little consideration will show that if P moves through a certain distance, say 1 foot, W will move through *half* that

distance, or 6 inches; hence, a pull of 1 pound at P will lift 2 pounds at W .

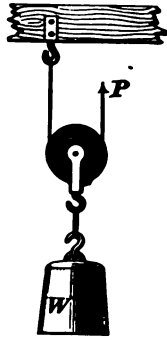


FIG. 146.

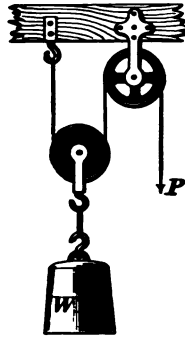


FIG. 147.

The same would also be true if the free end of the cord were passed over a *fixed pulley*, as in Fig. 147, in which case the fixed pulley merely changes the direction in which P acts, so that a weight of 1 pound hung on the free end of the cord will balance 2 pounds hung from the *movable pulley*.

938. A combination of pulleys, as shown in Fig. 148, is sometimes used. In this case there

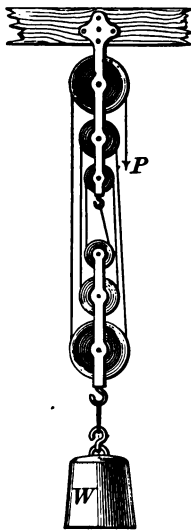


FIG. 148.

are three movable and three fixed pulleys, and the amount of movement of W , owing to a certain movement of P , is readily found.

It will be noticed that there are *six parts* of the rope, not counting the free end; hence, if the movable block be lifted 1 foot, P remaining in the same position, there would be 1 foot of slack in each of the six parts of the rope, or *six feet* in all. Therefore, P would have to move 6 feet in order to take up this slack, or P moves six times as far as W . Hence, 1 pound at P will support 6 pounds at W , since the *power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves*. It will also be noticed that there are three movable pulleys, and that $3 \times 2 = 6$.

Rule VIII.—*In any combination of pulleys where one continuous rope is used a load on the free end will balance a weight on the movable block as many times as great as the load on the free end as there are parts of the rope supporting the load—not counting the free end.*

The above law is good, whether the pulleys are side by side, as in the ordinary *block and tackle*, or whether they are arranged as in the figure.

EXAMPLE.—In a block and tackle having five movable pulleys, how great a force must be applied to the free end of the rope to raise 1,250 pounds?

SOLUTION.—Since there are 5 movable pulleys, there are 10 parts of the rope supporting them, and 1 pound on the free end will balance 10 pounds on the movable block; therefore, the ratio of P to W is 1 : 10, and $P = \frac{1,250}{10} = 125$ pounds. Ans.

939. In Fig. 149 is shown an arrangement called a **differential pulley**. It will be seen that if a force be applied at P , so as to pull the point P down to D , the rope or chain will wind up on the large pulley A , and unwind from the smaller pulley B , and since C is a movable pulley, the weight W will move an amount equal to one-half the difference between the amount of winding on A and unwinding on B .

Let the radius of A be represented by R , and of B by r ; then, when P is pulled down to D , a point E on the pulley A will move through a certain angle, $E O F$, the length of the arc $E F$ being equal to the distance $P D$. Any point on the pulley B , which is fastened to A , will turn through the same angle; and the difference between the arc $K L$, through which this point turns, and the arc $E F$ will be proportional to the difference of the radii R and r .

When P moves down to D , the point H on the other side will move up through the same distance to H' . The point I will move up to I' , a distance equal to the length

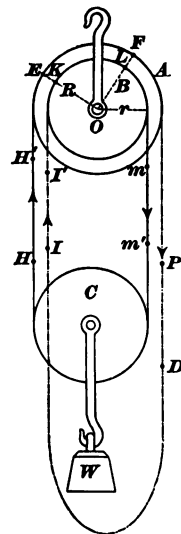


FIG. 149.

of the arc KL , and m will fall through the same distance to m' , thus causing the weight W to be *raised one-half the difference between mm' and PD* . The ratio between the distances through which W and P move will be

$$\frac{\text{arc } EF - \text{arc } KL}{2} : EF, \text{ or } \frac{R - r}{2} : R.$$

$$\text{Hence, } W \times \frac{R - r}{2} = P \times R, \text{ or } W = \frac{2PR}{R - r}. \quad (21.)$$

EXAMPLE.—If $R = 7$ inches, and $r = 6\frac{1}{4}$ inches, how much weight can be raised at W with a force of 50 pounds at P ?

$$\text{SOLUTION.}— W = \frac{2PR}{R - r} = \frac{2 \times 50 \times 7}{7 - 6\frac{1}{4}} = 1,400 \text{ pounds. Ans.}$$

THE INCLINED PLANE.

940. An **inclined plane** is a slope or a flat surface, making an angle with a horizontal line.

Three cases may arise in practice with the inclined plane:

- I. When the power acts parallel to the plane, as in Fig. 150.
- II. When the power acts parallel to the base, as in Fig. 151.
- III. When the power acts at an angle to the plane, or to the base, as in Fig. 152.

Case I.—In Fig. 150, the relation existing between the power and the weight is easily found. The weight ascends a distance equal to cb , or the height of the inclined plane,

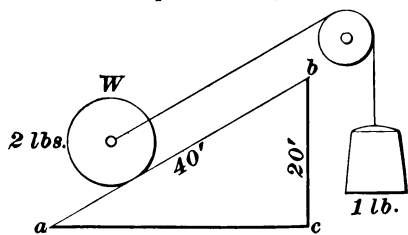


FIG. 150.

while the power descends through a distance equal to ab , or the length of the inclined plane. Therefore, the *power multiplied by the length of the inclined plane equals the weight multi-*

plied by the height of the inclined plane. Hence, if the length $ab = 40$ feet, and the height $cb = 20$ feet, $W \times 20 = P \times 40$, or 1 pound at P will balance 2 pounds at W .

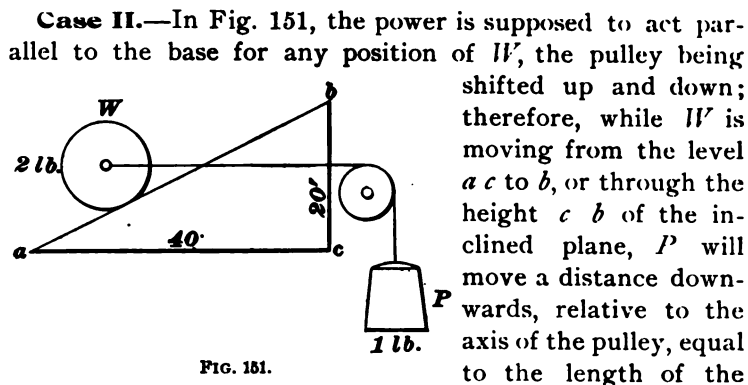


FIG. 151.

base ac . Hence, when the power acts parallel to the base, $W \times \text{height of the inclined plane} = P \times \text{length of base}$.

If the length of the base is 40 feet, and the height of the inclined plane is 20 feet, $W \times 20 = P \times 40$, and 1 pound at P will balance 2 pounds at W .

Case III.—For Fig. 152 no rule can be given. The ratio of the power to the weight must be determined for every position of W by means of the triangle or parallelogram of forces.

This case will be explained by means of an example, as it affords a splendid illustration of the principle of *resolution of forces*.

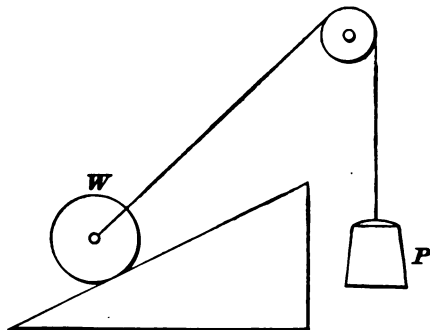


FIG. 152.

EXAMPLE.—In Fig. 153, a body W is shown resting on an inclined plane AB , whose dimensions are marked on the cut; the weight P acts to pull the body up the plane by means of the rope r and pulley p . It is required to find what the weight of P must be in order to start W up the plane. Suppose W weighs 120 pounds, and that friction is neglected. It is also required to find the perpendicular pressure which W exerts against the plane.

SOLUTION.—Through the point a , the center of gravity of W , draw ab vertical, and make it of such a length as to represent 120 pounds to a convenient scale, say 60 pounds = 1 inch. Drawing ac and cb ,

respectively, parallel and perpendicular to the plane, ac represents the magnitude of the force which must be exerted parallel to AB in order to put the body in equilibrium—i. e., to balance the force which gravity exerts in pulling the body down the plane. If the rope r were parallel to AB , ac would represent the weight of P ; but, since r makes an angle with the plane, P will not be equal to ac . To find what the weight of P must be, draw ad parallel to ac , but indicate it as acting in the opposite direction, or from a to d instead of from a to c . Now,

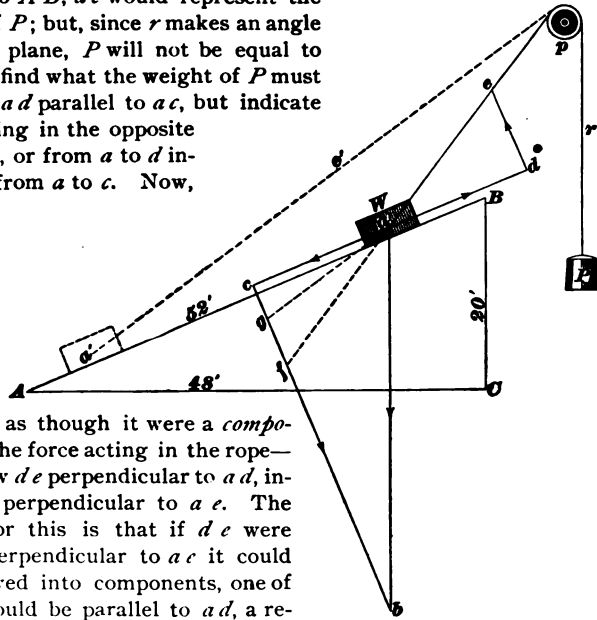


FIG. 153.

treat ad as though it were a *component* of the force acting in the rope—i. e., draw de perpendicular to ad , instead of perpendicular to ac . The reason for this is that if de were drawn perpendicular to ac it could be resolved into components, one of which would be parallel to ad , a result which we wish to avoid; in other

words, we want de perpendicular to the plane. The line ae , measured to the same scale as ab , will give the value of P . Measuring it, its length is .89"; hence, $P = .89 \times 60 = 53.4$ pounds. Ans.

To determine the perpendicular pressure against the plane, it will be noticed that ab equals the pressure due to gravity. Since cb and de are both perpendicular to AB , they are parallel, and since de acts in the opposite direction to cb , the actual pressure against the plane is given by the difference between cb and de . Making cf equal to de , fb represents the perpendicular pressure against the plane when the force $P (= ac)$ acts as shown. The length of fb is 1.39"; hence, the perpendicular pressure is $1.39 \times 60 = 83.4$ pounds. Ans.

Since ca and ad are parallel and equal, and cf and de are also parallel and equal, it follows that af and ae must also be parallel and equal. Consequently, the force P might have been found by drawing af parallel to the direction in which the pull on the rope acts, and bf perpendicular to the plane AB . Thus, suppose that the weight occupies the position shown by the dotted lines. Then, drawing ag

parallel to $a'e'$, ag represents the weight of P , and gb represents the perpendicular pressure of the body W' against the plane. Measuring ag , its length is .79"; hence, $P = .79 \times 60 = 47.4$ pounds. Measuring gb , its length is 1.65"; hence, the perpendicular pressure $= 1.65 \times 60 = 99$ pounds.

941. The Wedge.—

The **wedge** is a movable inclined plane, and is used for moving a great weight a short distance. A common method of moving a heavy body is shown in Fig. 154.

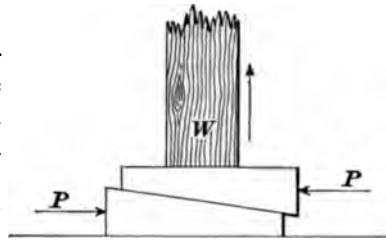


FIG. 154.

Simultaneous blows of equal force are struck on the heads of the wedges, thus raising the weight W' . The laws for wedges are the same as for Case II of the inclined plane.

THE SCREW.

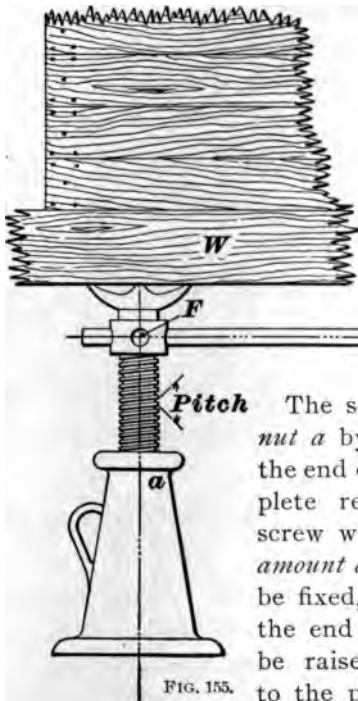


FIG. 155.

942. A screw

is a cylinder with a spiral groove winding around its circumference. This spiral is called the **thread** of the screw. The distance that the thread is drawn back or advanced in one turn of the screw is called the **pitch** of the screw.

The screw in Fig. 155 is turned in a nut a by means of a force applied at the end of the handle P . For one complete revolution of the handle, the screw will be advanced lengthwise an amount equal to the *pitch*. If the nut be fixed, and a weight be placed upon the end of the screw, as shown, it will be raised vertically a distance equal to the pitch by one revolution of the

screw. During this revolution the force at P will move through a distance equal to the circumference, whose radius is PF . Hence, $W \times \text{pitch of thread} = P \times \text{circumference of } P$.

943. Single-threaded screws of less than 1 inch pitch are generally classified by the number of threads they have in 1 inch of their length. In such cases, *one inch divided by the number of threads equals the pitch*; thus, the pitch of a screw that has 8 threads per inch is $\frac{1}{8}$ "; one of 32 threads per inch is $\frac{1}{32}$ ", etc.

EXAMPLE.—It is desired to raise a weight by means of a screw having 5 threads per inch. The force applied is forty pounds at a distance of 14 inches from the center of the screw; how great a weight can be raised?

SOLUTION.—Diameter of the circumference passed through by the force $= 14 \times 2 = 28$ inches. Therefore, $W \times \frac{1}{5} = 40 \times 28 \times 3.1416$, or $W = 17,593$ pounds. Ans.

VELOCITY RATIO.

944. The ratio of the distance that the power moves to the distance which the weight moves on account of the movement of the power is called the **velocity ratio**.

Thus, if the power is moving 12 inches while the weight is moving 1 inch, the velocity ratio is 12 to 1, or 12; that is, P moves 12 times as fast as W .

945. If the velocity ratio is known, the weight which any machine can raise equals the *power multiplied by the velocity ratio*. If the velocity ratio is 8.7 to 1, or 8.7, $W = 8.7 \times P$, since $W \times 1 = P \times 8.7$.

NOTE.—In all of the preceding cases, including the last, friction has been neglected.

FRICTION.

946. **Friction** is the resistance that a body meets from the surface on which it moves.

947. The **ratio** between the *resistance* to the motion of a body due to friction and the *perpendicular* pressure between the surfaces is called the **coefficient of friction**.

If a weight W , as in Fig. 156, rests upon a horizontal plane, and has a cord fastened to it passing over a pulley a , from which a weight P is suspended, then, if P is just sufficient to start W , the ratio of P to W , or $\frac{P}{W}$, is the *coefficient of friction* between W and the surface it slides upon.

The weight W is the perpendicular pressure, and P is the force necessary to overcome the resistance to the motion of W due to friction.

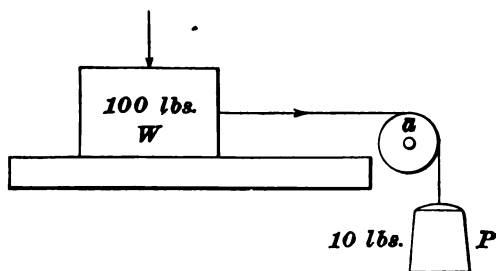


FIG. 156.

If $W = 100$ pounds and $P = 10$ pounds, the coefficient of friction for this particular case would be $\frac{P}{W} = \frac{10}{100} = .1$.

948. Laws of Friction :—

- I. *Friction is directly proportional to the perpendicular pressure between the two surfaces in contact.*
- II. *Friction is independent of the extent of the surfaces in contact when the total perpendicular pressure remains the same.*
- III. *Friction increases with the roughness of the surfaces.*
- IV. *Friction is greater between surfaces of the same material than between those of different materials.*
- V. *Friction is greatest at the beginning of motion.*
- VI. *Friction is greater between soft bodies than between hard ones.*
- VII. *Rolling friction is less than sliding friction.*
- VIII. *Friction is diminished by polishing or lubricating the surfaces.*

Law I shows why the friction is so much greater on journals after they begin to heat than before. The heat causes the journal to expand, thus increasing the pressure between the journal and its bearing, and, consequently, increasing the friction.

Law II states that, no matter how small the surface may be which presses against another, if the perpendicular pressure is the same, the friction will be the same. Therefore, large surfaces are used where possible; not to reduce the friction, but to reduce the wear and diminish the liability of heating.

For instance, if the perpendicular pressure between a journal and its bearing is 10,000 pounds, and the coefficient of friction is .2, the amount of friction is $10,000 \times .2 = 2,000$ pounds.

Suppose that one-half the area of the surface of the journal is 80 square inches; then, the amount of friction for each square inch of bearing is $2,000 \div 80 = 25$ pounds.

If half the area of the surface had been 160 square inches, the friction would have been the same—that is, 2,000 pounds; but the friction per square inch would have been $2,000 \div 160 = 12\frac{1}{2}$ pounds, just one-half as much as before, and the wear and liability to heat would be one-half as great also.

COEFFICIENTS OF FRICTION.

TABLE 17.

Description of Surfaces in Contact.	Disposition of Fibers.	State of the Surfaces.	Coefficient of Friction.
Oak on oak	Parallel	Dry	.48
Oak on oak	Parallel	Soaped	.16
Wrought iron on oak.....	Parallel	Dry	.62
Wrought iron on oak.....	Parallel	Soaped	.21
Cast iron on oak	Parallel	Dry	.49
Cast iron on oak	Parallel	Soaped	.19
Wrought iron on cast iron..	—	Slightly Unctuous	.18
Wrought iron on bronze....	—	Slightly Unctuous	.18
Cast iron on cast iron	—	Slightly Unctuous	.15

EFFICIENCY.

949. The force which is required to raise a weight, or overcome an equal resistance in any machine, is always *greater than this weight or resistance, divided by the velocity ratio of the machine.*

Thus, if there were no friction, a machine whose velocity ratio was 5 would, by an application of 100 pounds of force, raise a weight of 500 pounds.

Now, suppose that the friction in the machine is equivalent to 10 pounds of force, then it would take 110 pounds of force to raise 500 pounds.

If, in the above illustration, friction were neglected, 110 pounds $\times 5 = 550$ pounds, or the weight that 110 pounds would raise; but, owing to the frictional resistance, it only raised 500 pounds; therefore, we have for the ratio between the two $\frac{500}{550} = .91$. That is, $500 : 550 = .91 : 1$.

950. This ratio between the weight actually raised and the applied force multiplied by the velocity ratio is called the **efficiency of the machine.**

Let F = the force applied to the machine;

V = the velocity ratio of the machine;

W = the weight actually lifted, or equivalent resistance overcome;

E = the efficiency of the machine.

$$\text{Then, } E = \frac{W}{FV}. \quad (22.)$$

EXAMPLE.—In a machine having a combination of pulleys and gears, the velocity ratio of the whole is 9.75. A force of 250 pounds just lifts a weight of 1,626 pounds; what is the efficiency of the machine?

$$\text{SOLUTION.}—E = \frac{W}{FV} = \frac{1,626}{250 \times 9.75} = .6671, \text{ or } 66.71\%. \text{ Ans.}$$

Since the total amount of friction varies with the load, it follows that the efficiency will also vary for different loads.

EXAMPLE.—A pulley block with a velocity ratio of 7 has an efficiency of 38% for a load of 3,000 pounds; what power is required to start the load?

$$\text{SOLUTION.}—E = \frac{W}{FV}, \text{ or } .38 = \frac{3,000}{7F}$$

$$\text{Hence, } F = \frac{3,000}{.38 \times 7} = 1,128 \text{ pounds, nearly. Ans.}$$

EXAMPLES FOR PRACTICE.

1. A wedge is caused to move a weight vertically by means of a screw, which pulls the wedge horizontally on its base. If the screw has 5 threads per inch and the handle is 10 inches long, what force will be necessary to apply to the handle to raise a weight of 1,400 pounds, the height of the wedge being 8 inches, and the length, 14 inches?

Ans. 2.546 lb., nearly.

2. If the distance from the fulcrum to the point at which a force of 135 pounds is applied to a lever is 4 feet, and the distance from the fulcrum to the weight is $1\frac{1}{2}$ inches, how great a weight will the force lift?

Ans. 4,320 lb.

3. It is desired to raise a weight of 600 pounds by means of a block and tackle having two fixed and two movable pulleys; what force must be applied at the free end of the rope?

Ans. 150 lb.

4. If, in the last example, the free end of the rope be attached to a windlass whose drum is 5 inches in diameter, and which has a handle situated 15 inches from the axis of the drum, what force will be necessary to raise the weight?

Ans. 25 lb.

5. It is required to pull a wagon up an inclined plane one mile long. The height of the plane being 120 feet, and the weight of the wagon and load 2,816 pounds, what force will be necessary?

Ans. 64 lb.

6. In Fig. 144, the radius of the wheel *A* is 32 inches, of *C*, 18 inches, and of *E*, 30 inches; the radius of the drum *F* is 10 inches, of the pinion *D*, 4 inches, and of the pinion *B*, 6 inches. If *P* moves 4 feet 6 inches, how far will the weight *W* move?

Ans. $\frac{1}{4}$ in.

7. In the last example, how great a force must be applied at *P* to raise a weight of 2,160 pounds?

Ans. 30 lb.

8. In example 4, if the friction be taken as 22% of the load lifted, what force will be necessary? What will be the efficiency?

Ans. $\begin{cases} 30.5 \text{ lb.} \\ 82\%, \text{ nearly.} \end{cases}$

WORK AND ENERGY.

951. *Work is the overcoming of resistance continually occurring along the path of motion.*

Mere motion is not work, but if a body in motion constantly overcomes a resistance, it does work.

952. The **unit of work** is one pound raised vertically one foot, and is called one **foot-pound**. All work is measured by this standard. A horse going up hill does an amount of work equal to his own weight, plus the weight of the wagon and contents, plus the frictional resistances

reduced to an equivalent weight, multiplied by the vertical height of the hill. Thus, if the horse weighs 1,200 pounds, the wagon and contents 1,200 pounds, and the frictional resistances equal 400 pounds, then, if the vertical height of the hill is 100 feet, the work done is equal to $(1,200 + 1,200 + 400) \times 100 = 280,000$ foot-pounds.

953. If the force necessary to overcome the resistance be represented by F , the space through which the resistance acts by S , and the work done by U , then $U = F S$.

If W = the weight of a body, and h = the height through which it is raised, $U = W h$. Hence, the work done

$$U = F S = W h. \quad (23.)$$

954. The *total* amount of work is independent of time, whether it takes one minute or one year in which to do it; but, in order to compare the work done by different machines with a common standard, time must be considered.

When time is considered, the unit of time is always *one minute*, and the unit which measures the capacity of any contrivance for producing work then becomes one **foot-pound per minute**. This unit is called the **unit of power**. The term *power* as here used has a different meaning from that previously given to it, which simply meant force acting to produce motion in simple machines. The student should carefully distinguish between the two. There will be no difficulty in doing this, as the wording of the sentence in which it occurs will always show whether force or work per minute is meant. The power of a machine may always be determined by *dividing the work done in foot-pounds by the time in minutes required to do the work*; i. e.,

$$\text{Power} = \frac{FS}{T}, \quad (24.)$$

in which F and S have the same values as in formula **23**, and T = the time in minutes. Hence, if a certain machine does, say, 10,000 foot-pounds of work in 10 minutes, its *power* is $10,000 \div 10 = 1,000$ foot-pounds per minute; if another machine does the same work in 5 minutes, its power is $10,000 \div 5 = 2,000$ foot-pounds per minute—just twice as

much. Hence, we say that the power of the second machine is twice that of the first, and understand thereby that if both machines work for the same length of time the second machine will do twice as much work as the first machine.

955. Since the unit of power is very small, and would lead to the use of very large numbers in expressing the power of large machines, the common standard to which all work is reduced is the horsepower, which equals 33,000 units of power.

One horsepower is 33,000 foot-pounds per minute; in other words, it is 33,000 pounds raised vertically one foot in one minute, or 1 pound raised vertically 33,000 feet in one minute, or any combination that will give 33,000 foot-pounds in one minute by multiplying the resistance in pounds by the distance in feet through which it is overcome, and dividing by the time in minutes.

Thus, 110 pounds raised vertically 5 feet in one second, is a horsepower; for, since one second = $\frac{1}{60}$ of a minute, $110 \times 5 \div \frac{1}{60} = 33,000$ foot-pounds in one minute. The abbreviation for horsepower is H. P.

EXAMPLE.—If the coefficient of friction is .3, how many horsepower will it require to draw a load of 10,000 pounds on a level surface a distance of one mile in one hour?

SOLUTION.— $10,000 \times .3 = 3,000$ pounds = the force necessary to overcome the resistance (resistance of the air is neglected). One mile = 5,280 feet; one hour = 60 minutes.

$$\text{Power} = \frac{FS}{T} = \frac{3,000 \times 5,280}{60} = 264,000 \text{ foot-pounds per minute.}$$

$$\text{Horsepower} = \frac{264,000}{33,000} = 8 \text{ H.P. Ans.}$$

956. Energy is a term used to express *the ability of an agent to do work*. Work cannot be done without motion, and the work that a moving body is capable of doing in being brought to rest is called the **kinetic energy** of the body.

Kinetic energy means the actual energy of a body in motion. The work which a moving body is capable of doing in being brought to rest is exactly the same as the kinetic

energy developed by it when falling in a vacuum through a height sufficient to give it the same velocity.

Let W = the weight of the body in pounds;

v = its velocity in feet per second;

h = the height in feet through which the body must fall to produce the velocity v ;

m = the mass of the body = $\frac{W}{g}$. (See formula 10,

Art. 888.)

957. The work necessary to raise a body through a height h is Wh . The velocity produced in falling through a height h is $v = \sqrt{2gh}$, and $h = \frac{v^2}{2g}$. (See formulas 15 and 16, Art. 896.)

Therefore, work = $Wh = W \frac{v^2}{2g} = \frac{1}{2} \times \frac{W}{g} \times v^2 = \frac{1}{2} m v^2$, or

$$Wh = \frac{1}{2} m v^2. \quad (25.)$$

In other words, if the weight of the body and its velocity are given, the work necessary to bring it to rest is equal to one-half the product of the mass and the square of the velocity in feet per second. This is the kinetic energy of a moving body.

If a body, whose weight is 64.32 pounds, is moving with a velocity of 20 feet per second, the kinetic energy is $\frac{1}{2} m v^2 = \frac{W}{2g} \times v^2 = \frac{64.32}{2 \times 32.16} \times 20^2 = 400$ foot-pounds.

958. The height through which a body would have to fall in a vacuum to gain a velocity of 20 feet per second is $h = \frac{v^2}{2g} = \frac{400}{2 \times 32.16} = \frac{400}{64.32}$; the work done in raising a body through this height is $Wh = 64.32 \times \frac{400}{64.32} = 400$ foot-pounds, the same result as before. The distinction between kinetic energy and work, then, is this: *A body moving with a certain velocity, and meeting with no resistance, is not doing any work, but has a certain amount of kinetic energy stored up in it, which depends upon the velocity and mass of the body.*

If a resistance be interposed just sufficient to stop the body, the body will then do an amount of *work* equal to its kinetic energy.

EXAMPLE.—If a body weighing 25 pounds falls from a height of 100 feet, how much work can be done?

SOLUTION.—Work = $Wh = 25 \times 100 = 2,500$ foot-pounds. Ans.

EXAMPLE.—A body weighing 50 pounds has a velocity of 100 feet per second; what is its kinetic energy?

SOLUTION.—Kinetic energy = $\frac{1}{2}mv^2 = \frac{Wv^2}{2g} = \frac{50 \times 100^2}{2 \times 32.16} = 7,773.63$ foot-pounds. Ans.

EXAMPLE.—In the last example, how many horsepower would be required to give the body this amount of kinetic energy in 3 seconds?

SOLUTION.—1 H.P. = 33,000 pounds raised one foot in one minute. If 7,773.63 foot-pounds of work are done in 3 seconds, in one second there would be done $\frac{7,773.63}{3} = 2,591.21$ foot-pounds of work, and in one minute, $2,591.21 \times 60 = 155,472.6$ foot-pounds.

The number of horsepower developed would be

$$\text{H.P.} = \frac{155,472.6}{33,000} = 4.7113 \text{ H.P. Ans.}$$

959. Potential energy is latent energy; it is the energy which a body at rest is capable of giving out under certain conditions.

If a stone be suspended by a string from a high tower, it has potential energy due to its position. If the string be cut, the stone will fall to the ground, and during its fall its potential energy will change into kinetic energy, so that, at the instant it strikes the ground, its potential energy is wholly changed into kinetic energy.

At a point equal to one-half the height of the fall, the potential and kinetic energies were equal. At the end of the first quarter, the potential energy was $\frac{3}{4}$, and the kinetic energy $\frac{1}{4}$; at the end of the third quarter, the potential energy was $\frac{1}{4}$, and the kinetic energy $\frac{3}{4}$.

A pound of coal has a certain amount of potential energy. When the coal is burned, the potential energy is liberated, and changed into kinetic energy in the form of heat. The kinetic energy of the heat changes water into steam, which

thus has a certain amount of potential energy. The steam acting on the piston of an engine causes it to move through a certain space, thus overcoming a resistance, changing the potential energy of the steam into kinetic energy, and thus doing work.

Potential energy, then, is the energy stored within a body, which may be liberated and produce motion, thus generating kinetic energy, and enabling work to be done.

960. The principle of **conservation of energy** teaches that energy, like matter, can never be destroyed. If a clock is put in motion, the potential energy of the spring is changed into the kinetic energy of motion, which turns the wheels, thus producing friction. The friction produces heat, which dissipates into the surrounding air, but still the energy is not destroyed—it merely exists in another form. The potential energy in coal was received from the sun, in the form of heat, ages ago, and has laid dormant for millions of years.

FORCE OF A BLOW.

961. The *average force of a blow* may be determined as follows:

In driving a nail into a piece of wood with a hammer, the head of the hammer must be capable of exerting, in a very short time, a force equal to that of a load sufficiently heavy to produce by its weight a movement of the nail into the wood, equal to the movement of the nail produced by the hammer; in other words, *the striking force, multiplied by the distance that the nail is driven into the wood, must equal the kinetic energy of the hammer.*

Suppose that the velocity of the hammer, as it strikes the nail, is 30 feet per second, that the weight of the head is 2 pounds, and that the nail is driven into the wood $\frac{1}{4}$ of an inch. Let the resistance offered by the wood, which is the same as the striking force of the hammer, be represented by F ; then, since $\frac{1}{4}$ inch = $\frac{1}{48}$ of a foot,

$$F \times \frac{1}{48} = \frac{1}{2} m \times v^2 = \frac{2}{2 \times 32.16} \times 30^2 = 28, \text{ nearly.}$$

Therefore, $F = 28 \times 48 = 1,344$ pounds.

Had the penetration been $\frac{1}{8}$ of an inch, instead of $\frac{1}{4}$ of an inch, the value of F would have been twice as large, or 2,688 pounds. This is as it should be, since the work done is the same in both cases, while the distance through which the force acts is only half as great in the second case as in the first case. Hence, in order that the work may be the same, the force must be doubled.

EXAMPLE.—If the head of a drop-hammer, weighing 400 pounds, falls from a height of 10 feet, and compresses a piece of cold iron .01 of an inch, what was the striking force of the hammer?

SOLUTION.—.01 of an inch = $\frac{1}{12}$ of a foot.

$$F \times \frac{1}{12} = Wh = 400 \times 10 = 4,000.$$

$$\text{Hence, } F = \frac{4,000 \times 12}{.01} = 4,800,000 \text{ lb. Ans.}$$

DENSITY AND SPECIFIC GRAVITY.

962. The **density of a body** is its mass divided by its volume in cubic feet.*

Let D be the density; then, the density of a body is,

$$D = \frac{m}{V}. \quad \text{Since } m = \frac{W}{g}, \quad D = \frac{W}{gV}. \quad (26.)$$

EXAMPLE.—Three cubic feet of cast iron weigh 1,350 pounds; what is the density of cast iron?

$$\text{SOLUTION.}—D = \frac{W}{gV} = \frac{1,350}{32.16 \times 3} = 13.992.$$

EXAMPLE.—A cubic foot of water weighs 62.42 pounds; (a) what is its density? (b) What is the ratio between the density of cast iron and the density of water?

$$\text{SOLUTION.}—(a) \quad D = \frac{W}{gV} = \frac{62.42}{32.16 \times 1} = 1.941 = \text{density. Ans.}$$

$$(b) \quad \frac{13.992}{1.941} = 7.21 = \text{ratio. Ans.}$$

963. The **specific gravity of a body** is the ratio between its weight and the weight of a like volume of water.

***NOTE.**—Some writers define density as the weight of a unit of volume of the material. When English measures are used, the density of any material, according to this definition, is the weight of a cubic foot of the material in pounds.

964. Since gases are so much lighter than water, it is usual to take the specific gravity of a gas as the ratio between the weight of a certain volume of the gas and the weight of the same volume of air.

EXAMPLE.—A cubic foot of cast iron weighs 450 pounds; what is its specific gravity, a cubic foot of water weighing 62.42 pounds?

SOLUTION.—According to the definition,

$$\frac{450}{62.42} = 7.21. \quad \text{Ans.}$$

NOTE.—Notice that this answer is the same as that of the preceding example; hence, the specific gravity of a body is also the *ratio of the density of a body to the density of water*.

965. The specific gravities of different bodies are given in printed tables; hence, if it is desired to know the weight of a body that cannot be conveniently weighed, *calculate its cubical contents, and multiply the specific gravity of the body by the weight of a like volume of water, remembering that a cubic foot of water weighs 62.42 pounds.*

EXAMPLE.—How much will 3,214 cubic inches of cast iron weigh? Take its specific gravity as 7.21.

SOLUTION.—Since 1 cubic foot of water weighs 62.42 pounds, 3,214 cubic inches weigh

$$\frac{3,214}{1,728} \times 62.42 = 116.098 \text{ pounds.}$$

Then, $116.098 \times 7.21 = 837.067$ pounds. Ans.

EXAMPLE.—What is the weight of a cubic inch of cast iron?

SOLUTION.— $\frac{62.42}{1,728} \times 7.21 = .26044$ pound. Ans.

One cubic foot of pure distilled water at a temperature of 39.2° Fahrenheit weighs 62.425 pounds, but *the value usually taken in making calculations is 62.5 pounds.*

EXAMPLE.—What is the weight in pounds of 7 cubic feet of oxygen?

SOLUTION.—One cubic foot of air weighs .08073 pound, and the specific gravity of oxygen is 1.1056, compared with air; hence,
 $.08073 \times 1.1056 \times 7 = .62479$ pound, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. A man jumps from a railroad train traveling at the rate of 60 miles per hour; if he weighs 160 lb., what is the kinetic energy of his body?
 Ans. 19,263.68 ft.-lb.

2. A hammer strikes a nail with a velocity of 40 ft. per sec. ; if the head weighs $1\frac{1}{2}$ lb., and the nail is driven $\frac{1}{4}$ in., what is the force of the blow ?

Ans. 1,433 lb., nearly.

3. A load of 20,000 lb. is pulled up an inclined plane $1\frac{1}{2}$ miles long in 5 minutes; if the height of the plane is 800 ft., and the frictional resistances are 3% of the load, what is the horsepower required ?

Ans. 125.62 H. P.

4. If a cubic foot of a certain substance weighs $162\frac{1}{2}$ lb., what is its specific gravity ?

Ans. 2.6.

5. A mixture of lead and tin has a specific gravity of 9.28; what is the weight of a cubic inch ?

Ans. 5.37 oz.

6. A weight of 11,625 lb. is raised vertically 10 ft. in 3 minutes, by means of a block and tackle and a windlass; the frictional resistances being 26%, how many horsepower were required ?

Ans. 1.48 H. P.

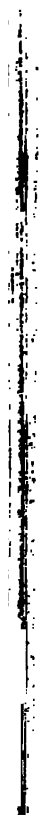
7. A body weighing 24,062.5 lb. is drawn on a horizontal surface at the rate of 600 ft. per min.; the coefficient of friction being 8%, what horsepower will be necessary to overcome the resistance of friction ?

Ans. 35 H. P.

8. A solid cast iron sphere, 12" in diameter, falls through a height of 50 ft. ; what will be its kinetic energy on striking ?

Ans. 11,781 ft.-lb.

966. The table of specific gravities gives the specific gravities of a variety of substances likely to be met with in ordinary practice. The weights per cubic foot are calculated on a basis of 62.5 pounds of water per cubic foot.



A SERIES
OF
QUESTIONS AND EXAMPLES
RELATING TO THE SUBJECTS
TREATED OF IN VOL. I.

It will be noticed that, although the various questions are numbered in sequence from **1** to **283** and from **355** to **453**, inclusive, these questions are divided into six different sections, corresponding to the six sections of the preceding pages of this volume. Under the heading of each section are given, in parenthesis, the numbers of those articles which should be carefully studied before attempting to answer any question or to solve any example occurring in the section.



ARITHMETIC.

(SEE ARTS. 1 to 174.)

- (1) What is arithmetic ?
- (2) What is a number ?
- (3) What is the difference between a concrete number and an abstract number ?
- (4) Define notation and numeration.
- (5) Write each of the following numbers in words:
980; 605; 28,284; 9,006,042; 850,317,002; 700,004.
- (6) Represent in figures the following expressions:
Seven thousand, six hundred. Eighty-one thousand, four hundred, two. Five million, four thousand, seven. One hundred eight million, ten thousand, one. Eighteen million, six. Thirty thousand, ten.
- (7) What is a fraction ?
- (8) What are the terms of a fraction ?
- (9) What does the denominator show ?
- (10) What does the numerator show ?
- (11) How do you find the value of a fraction ?
- (12) Is $1\frac{3}{8}$ a proper or an improper fraction and why ?
- (13) Write three mixed numbers.
- (14) Reduce the following fractions to their lowest terms:
 $\frac{4}{8}, \frac{4}{16}, \frac{8}{32}, \frac{3}{6}$. Ans. $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$.
- (15) Reduce 6 to an improper fraction whose denominator is 4. Ans. $\frac{24}{4}$.
- (16) Reduce $7\frac{7}{8}, 13\frac{5}{16}$, and $10\frac{3}{4}$ to improper fractions.
Ans. $\frac{59}{8}, \frac{213}{16}, \frac{43}{4}$.

(17) What is the value of each of the following: $\frac{13}{2}$, $\frac{11}{4}$, $\frac{11}{8}$, $\frac{16}{8}$, $\frac{11}{4}$? Ans. $6\frac{1}{2}$, $4\frac{1}{4}$, $4\frac{1}{8}$, 2, $1\frac{3}{4}$.

(18) State the difference between a fraction and a decimal.

(19) State how to reduce a fraction to a decimal.

(20) Reduce the following fractions to equivalent decimals: $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{8}$, $\frac{66}{100}$, and $\frac{125}{1000}$.

Ans. $\left\{ \begin{array}{l} .5. \\ .875. \\ .15625. \\ .65. \\ .125. \end{array} \right.$

(21) Write out in words the following numbers: .08, .131, .0001, .000027, .0108, and 93.0101.

(22) How do you place decimals for addition and subtraction?

(23) Give a rule for multiplication of decimals.

(24) Give a rule for division of decimals.

(25) What is the sum of 3,290, 504, 865,403, 2,074, 81, and 7? Ans. 871,359.

(26) $709 + 8,304,725 + 391 + 100,302 + 300 + 909 =$ what? Ans. 8,407,336.

(27) $\frac{1}{8} + \frac{2}{8} + \frac{5}{8} = ?$ Ans. 1.

(28) $\frac{1}{4} + \frac{3}{8} + \frac{5}{8} = ?$ Ans. $1\frac{5}{8}$.

(29) $42 + 31\frac{5}{8} + 9\frac{1}{8} = ?$ Ans. $83\frac{1}{8}$.

(30) An iron plate is divided into four sections; the first contains $29\frac{3}{4}$ square inches; the second, $50\frac{3}{8}$ square inches; the third, 41 square inches, and the fourth, $69\frac{3}{8}$ square inches. How many square inches are in the plate?

Ans. $190\frac{9}{8}$ sq. in.

(31) What is the sum of .125, .7, .089, .4005, .9, and .000027? Ans. 2.214527.

(32) $927.416 + 8.274 + 372.6 + 62.07938 = ?$ Ans. 1,370.36938.

(33) Add 17 thousandths, 2 tenths, and 47 millionths. Ans. .217047.

(34) The weights of four lengths of shafting are 316.5, 402.3, $348\frac{1}{2}$, and 309.4 pounds, respectively. What is their total weight? Ans. 1,376.95 lb.

(35) The cost of the coal consumed under a nest of steam boilers during a week's run was as follows: On Monday, \$15.83; Tuesday, \$14.70; Wednesday, \$14.28; Thursday, \$13.87; Friday, \$14.98; Saturday, \$12.65. What was the cost of the week's supply of coal? Ans. \$86.31.

(36) During a 12-hour test upon a steam engine the counter showed the number of revolutions per hour to have been as follows: 12,600, 12,444, 12,467, 12,528, 12,468, 12,590, 12,610, 12,589, 12,576, 12,558, 12,546, 12,532. How many revolutions were made during the test? Ans. 150,508 revolutions.

(37) Find the difference between the following:

(a) 50,962 and 3,338; (b) 10,001 and 15,339.

Ans. $\begin{cases} (a) 47,624. \\ (b) 5,338. \end{cases}$

(38) (a) $70,968 - 32,975 = ?$ (b) $100,000 - 98,735 = ?$

Ans. $\begin{cases} (a) 37,993. \\ (b) 1,265. \end{cases}$

(39) What is the difference between $\frac{7}{8}$ and $\frac{1}{16}$, 13 and $7\frac{1}{16}$, $312\frac{2}{16}$ and $229\frac{5}{16}$? Ans. $\frac{1}{16}$, $5\frac{1}{16}$, $83\frac{1}{16}$.

(40) Work out the following examples:

(a) $709.63 - .8514$; (b) $81.963 - 1.7$; (c) $18 - .18$;
(d) $1 - .001$; (e) $872.1 - (.8721 + .008)$; (f) $(5.028 + .0073) - (6.704 - 2.38)$.

Ans. $\begin{cases} (a) 708.7783. \\ (b) 80.263. \\ (c) 17.82. \\ (d) .999. \\ (e) 871.2199. \\ (f) .7113. \end{cases}$

(41) Work out the following:

(a) $\frac{1}{8} - .807$; (b) $.875 - \frac{3}{8}$; (c) $(\frac{5}{8} + .435) - (\frac{21}{100} - .07)$.
(d) What is the difference between the sum of 33 millionths

and 17 thousandths, and the sum of 53 hundredths and 274 thousandths?

$$\text{Ans. } \begin{cases} (a) .068. \\ (b) .5. \\ (c) .45125. \\ (d) .786967. \end{cases}$$

(42) It is desired to increase the capacity of an electric-light plant to 1,500 horsepower by adding a new engine. If the indicated horsepower of the engines already in use is $482\frac{1}{2}$, $316\frac{1}{2}$, and $390\frac{1}{2}$, what power must the new engine develop?

Ans. $310.11\frac{1}{2}$ H. P.

(43) On a certain morning 7,240 gallons of water were drawn from an engine-room tank, and 4,780 gallons were pumped in. In the afternoon 7,633 gallons were drawn out, and 8,675 gallons pumped in. How many gallons remained in the tank at night, if it contained 3,040 gallons at the beginning of the day?

Ans. 1,622 gal.

(44) A metal stack 45 feet high is made up of 7 plates, 6 of which are 7 feet long. Allowing 15 inches for laps, what is the length of the seventh plate?

Ans. 4 ft. 3 in.

(45) The inside diameter of a 6-inch steam pipe is 6.06 inches, and the outside diameter is 6.62 inches. How thick is the pipe?

Ans. .28 in.

(46) Find the products of the following:

(a) $526,387 \times 7$; (b) $700,298 \times 17$; (c) $217 \times 103 \times 67$.

$$\text{Ans. } \begin{cases} (a) 3,684,709. \\ (b) 11,905,066. \\ (c) 1,497,517. \end{cases}$$

(47) If your watch ticks once every second, how many times will it tick in one week?

Ans. 604,800.

(48) Find the products of the following expressions:

(a) $.013 \times .107$; (b) $203 \times 2.03 \times .203$; (c) $(2.7 \times 31.85) \times (3.16 - .316)$; (d) $(107.8 + 6.541 - 31.96) \times 1.742$.

$$\text{Ans. } \begin{cases} (a) .001391. \\ (b) 83.65427. \\ (c) 244.56978. \\ (d) 143.507702. \end{cases}$$

(49) An engine and boiler in a manufactory are worth \$3,246. The building is worth three times as much plus \$1,200, and the tools are worth twice as much as the building plus \$1,875; (a) what is the value of the building and tools? (b) What is the value of the whole plant?

Ans. $\begin{cases} (a) \$34,689. \\ (b) \$37,935. \end{cases}$

(50) How many square feet of heating surface are in the tubes of a boiler having 60 3-inch tubes, each $15\frac{1}{2}$ feet long, if the heating surface of each tube per foot in length is .728 square foot?

Ans. 677.04 sq. ft.

(51) Suppose that in one hour 10 pounds of coal are burned per square foot of grate area in a certain boiler, and that 9 pounds of water are evaporated per pound of coal burned. If the grate area is 30 square feet, how many pounds of water would be evaporated in a day of 10 hours?

Ans. 27,000 lb.

(52) How many feet does the piston of a steam engine pass over in a week of 6 days, running $8\frac{1}{2}$ hours a day, if the length of the stroke of the engine is $1\frac{1}{2}$ feet, and the number of revolutions per minute 160?

Ans. 1,468,800 ft.

(53) A number of boilers are constantly fed by 3 pumps; the first delivers $1\frac{1}{4}$ gallons per stroke, and runs at 75 strokes per minute; the second delivers $\frac{2}{3}$ of a gallon per stroke, and runs at 115 strokes per minute; the third delivers $1\frac{3}{4}$ gallons per stroke, and runs at 96 strokes per minute. How many gallons of water are fed to the boilers per hour?

Ans. 21,022.5 gal.

NOTE.—If in the following examples there be a remainder, carry the quotient to four decimal places.

(54) Divide the following:

(a) 962,842 by 84; (b) 39,728 by 63; (c) 29,714 by 108;
(d) 406,089 by 135.

Ans. $\begin{cases} (a) 11,462.4048. \\ (b) 630.6032. \\ (c) 275.1296. \\ (d) 3,008.0667. \end{cases}$

(55) Solve the following:

(a) $35 \div \frac{5}{16}$; (b) $\frac{9}{16} \div 3$; (c) $\frac{17}{2} \div 9$; (d) $\frac{113}{64} \div \frac{7}{16}$; (e) $15\frac{3}{4} \div 4\frac{3}{8}$.

Ans. $\left\{ \begin{array}{l} (a) 112. \\ (b) \frac{3}{16}. \\ (c) \frac{17}{18}. \\ (d) 4\frac{1}{28}. \\ (e) 3\frac{3}{8}. \end{array} \right.$

(56) Solve the following:

(a) $.875 \div \frac{1}{2}$; (b) $\frac{7}{8} \div .5$; (c) $\frac{.375 \times \frac{1}{4}}{\frac{5}{16} - .125}$.

Ans. $\left\{ \begin{array}{l} (a) 1.75. \\ (b) 1.75. \\ (c) .5. \end{array} \right.$

(57) Solve the following by cancelation:

(a) $(72 \times 48 \times 28 \times 5) \div (84 \times 15 \times 7 \times 6)$.

(b) $(80 \times 60 \times 50 \times 16 \times 14) \div (70 \times 50 \times 24 \times 20)$.

Ans. $\left\{ \begin{array}{l} (a) 94. \\ (b) 32. \end{array} \right.$

(58) Find the values of the following expressions:

(a) $\frac{7}{\frac{3}{16}}$; (b) $\frac{\frac{15}{2}}{\frac{3}{8}}$; (c) $\frac{1.25 \times 20 \times 3}{\frac{87 + 88}{459 + 32}}$.

Ans. $\left\{ \begin{array}{l} (a) 37\frac{1}{2}. \\ (b) .75. \\ (c) 210\frac{3}{4}. \end{array} \right.$

(59) The distance around a cylindrical boiler is 166.85 inches. If there are 72 rivets in one of the circular seams, find what the pitch (distance between the centers of any two rivets) of the rivets is.

Ans. $2.317 + \text{in.}$

(60) A keg of $\frac{7}{8} \times 2\frac{3}{4}$ inches boiler rivets weighs 100 pounds, and contains 133 rivets. What is the weight of each rivet?

Ans. $.75 + \text{lb.}$

(61) The distance around a wheel equals 3.1416 times its diameter, or the distance across it. If the distance around a fly-wheel is 56.5488 feet, what is the diameter of a wheel half as large?

Ans. 9 ft.

(62) If 980,000 bricks are required to build an engine house, how many days will it take for 6 teams to draw them, each team drawing 8 loads a day, and there being 1,600 bricks to a load?

Ans. $12.76 + \text{days.}$

(63) If a mechanic earns \$1,500 a year for his labor, and his expenses are \$968 per year, in what time can he save enough to buy 28 acres of land at \$133 an acre?

Ans. 7 years.

(64) The numerator of a fraction is 28, and the value of the fraction $\frac{7}{8}$; what is the denominator? Ans. 32.

(65) From 1 plus .001 take .01 plus .000001.

Ans. .990999.

(66) A freight train ran 365 miles in one week, and 3 times as far, lacking 246 miles, the next week; how far did it run the second week? Ans. 849 miles.

(67) If the driving wheel of a locomotive is 16 ft. in circumference, how many revolutions will it make in going from Philadelphia to Pittsburg, the distance of which is 354 miles, there being 5,280 feet in one mile?

Ans. 116,820 rev.

(68) How many inches in .875 of a foot? Ans. $10\frac{1}{2}$ in.

(69) What decimal part of a foot is $\frac{3}{16}$ of an inch?

Ans. .015625 ft.

(70) If water be conducted from a tank by 7 lengths of gas pipe coupled together, each length being 12 feet 6 inches long, $\frac{3}{4}$ of an inch being added at each of the joints for coupling together, how far from the tank is the water discharged? The first length screws into the tank $\frac{3}{4}$ of an inch. Ans. 87.8125 ft.

(71) If by selling a carload of coal for \$82.50, at a profit of \$1.65 per ton, I make enough to pay for 72.6 ft. of fencing at \$.50 a foot, how many tons of coal were there in the car? Ans. 22 tons.

(72) The connection between an engine and boiler is made up of 6 lengths of pipe, three of which are 14 feet 5 inches long, two 12 feet 6 inches long, and one 8 feet 10 inches long. If the pipe weighs $10\frac{1}{2}$ pounds per foot, what is the total weight of the pipe used? Ans. 809.375 lb.

(73) Four bolts are required, $2\frac{1}{2}$, $6\frac{7}{8}$, $3\frac{1}{16}$, and 4 inches long. How long a piece of iron will be required from

which to cut them, allowing $\frac{7}{16}$ of an inch to each bolt for cutting off and finishing?

Ans. $18\frac{3}{16}$.

(74) A double belt of a certain width can transmit $64\frac{1}{2}$ horsepower. How many horsepower can two single belts of the same width transmit, when running under the same conditions, supposing that the double belt is capable of transmitting $\frac{1}{2}$ as much power as one of the single belts?

Ans. 90.3 H. P.

(75) The lengths of belting required to connect four countershafts with the main line shaft were 18 feet 6 inches, 16 feet $9\frac{1}{2}$ inches, 22 feet 2 inches, and 20 feet $8\frac{1}{2}$ inches. How many feet of belting were required?

Ans. $78\frac{1}{2}$ ft.

(76) In a steam-engine test of an hour's duration, the horsepower developed was found to be as follows, at 10-minute intervals: 48.63, 45.7, 46.32, 47.9, 48.74, 48.38, 48.59. What was the average horsepower?

Ans. $47.75 +$ H. P.

ARITHMETIC.

(SEE ARTS. 175-339.)

- (77) What is 25% of 8,428 lb. ? Ans. 2,107 lb.
- (78) What is 1% of \$100 ? Ans. \$1.
- (79) What is $\frac{1}{2}$ % of \$35,000 ? Ans. \$175.
- (80) What per cent. of 50 is 2 ? Ans. 4%.
- (81) What per cent. of 10 is 10 ? Ans. 100%.
- (82) Solve the following:
- (a) Base = \$2,522, and percentage = \$176.54. What is the rate ?
- (b) Percentage = 16.96, and rate = 8%. What is the base ?
- (c) Amount = 216.7025, and base = 213.5. What is the rate ?
- (d) Difference = 201.825, and base = 207. What is the rate ?
- Ans. $\left\{ \begin{array}{l} (a) \ 7\%. \\ (b) \ 212. \\ (c) \ 1\frac{1}{2}\%. \\ (d) \ 2\frac{1}{2}\%. \end{array} \right.$
- (83) The coal consumption of a steam plant is 5,500 lb. per day when the condenser is not running, or an increase of 15% over the consumption when the condenser is used. How many pounds are used per day when the condenser is running ? Ans. 4,782.61 lb.
- (84) An engineer receives a salary of \$950. He pays 24% of it for board, $12\frac{1}{2}$ % of it for clothing, and 17% of it for other expenses. How much does he save a year? Ans. \$441.75.
- (85) If $37\frac{1}{2}$ % of a number is 961.38, what is the number ? Ans. 2,563.68.

(86) A man owns $\frac{3}{4}$ of a manufacturing plant. 30% of his share is worth \$1,125. What is the whole property worth?

Ans. \$5,000.

(87) What number diminished by 35% of itself equals 4,810?

Ans. 7,400.

(88) The volume of the clearance in a steam-engine cylinder is found to be 18.3 cu. in., and the volume of the cylinder, neglecting the clearance, 254.5 cu. in. What percentage of the cylinder volume is the clearance?

Ans. 7.2%, nearly.

(89) The distance between two stations on a certain railroad is 16.5 miles, which is $12\frac{1}{2}\%$ of the entire length of the road. What is the length of the road?

Ans. 132 miles.

(90) The speed of an engine running unloaded was $1\frac{1}{2}\%$ greater than when running loaded. If it made 298 revolutions per minute with the load, what was its speed running unloaded?

Ans. 302.47 revolutions per minute.

(91) Reduce 4 yd. 2 ft. 10 in. to inches.

Ans. 178 in.

(92) Reduce 3,722 in. to higher denominations.

Ans. 103 yd. 1 ft. 2 in.

(93) How many seconds in 5 weeks and 3.5 days?

Ans. 3,326,400 sec.

(94) Reduce 764,325 cu. in. to cu. yd.

Ans. 16 cu. yd. 10 cu. ft. 549 cu. in.

(95) How many gallons of water can be put into a tank holding 4 bbl. 10 gal. 3 qt.?

Ans. 136 $\frac{1}{2}$ gal.

(96) A carload of coal weighed 16 T. 8 cwt. 75 lb. How many pounds did this amount to?

Ans. 32,875 lb.

(97) Reduce 25,396 lb. to higher denominations.

Ans. 12 T. 13 cwt. 96 lb.

(98) Reduce 25,396 pt. to higher denominations.

Ans. 100 bbl. 24 gal. 2 qt.

(99) What is the sum of 2 yd. 2 ft. 3 in.; 4 yd. 1 ft. 9 in.; 2 ft. 7 in.?

Ans. 8 yd. 7 in.

(100) What is the sum of 3 gal. 3 qt. 1 pt.; 6 gal. 1 pt.; 4 gal. 8 qt. 5 pt. ? Ans. 16 gal. 2 qt. 1 pt.

(101) From 52 yd. 2 ft. 9 in. take 115 ft. Ans. 14 yd. 1 ft. 9 in.

(102) From a barrel of machine oil is sold at one time 10 gal. 2 qt. 1 pt., and at another time 16 gal. 3 qt. How much remained ? Ans. 4 gal. 1 pt.

(103) If 1 iron rail is 17 ft. 3 in. long, how long would 51 such rails be, if placed end to end ? Ans. 879 ft. 9 in.

(104) Multiply 3 qt. 1 pt. by 4.7. Ans. 32.9 pt.

(105) How many iron rails, each 30 ft. long, are required to lay a single railroad track 23 miles long ? Ans. 8,096 rails.

(106) The main line shaft of a mill is composed of 4 lengths each 15 ft. 5 in. long, of one piece 14 ft. 8 in. long, and one piece 8 ft. 10 in. long. If there are 6 hangers spaced equally distant apart, one being placed at each extremity of the shaft, what is the distance between the hangers ? Ans. 17 ft. $\frac{2}{3}$ in.

(107) The distance around a wheel is approximately $\frac{22}{7}$ times the diameter. If the diameter of a fly-wheel is 9 ft. $6\frac{1}{2}$ in., find the distance around it. Ans. 29 ft. $11\frac{1}{4}$ in.

(108) If the length of a boiler shell is 18 ft. $11\frac{1}{4}$ in., how many rivets should there be in one of the longitudinal seams if it is a single-riveted seam, supposing the rivets to have a pitch of $1\frac{1}{4}$ in., and the two end rivets to be $1\frac{1}{4}$ in. from each end of the boiler ? Ans. 181 rivets.

(109)

(a) What is the second power of 108 ?

(b) What is the third power of 181.25 ?

(c) What is the fourth power of 27.61 ?

Ans. $\left\{ \begin{array}{l} (a) 11,664. \\ (b) 5,954,345.703125. \\ (c) 581,119.73780641. \end{array} \right.$

(110) Solve the following:

(a) 106^2 ; (b) $(182\frac{1}{8})^2$; (c) $.005^2$; (d) $.0063^2$; (e) 10.06^2 ,
(f) 67.85^2 ; (g) $967,845^2$; (h) $(\frac{1}{16})^2$; (i) $(\frac{1}{4})^2$.

Ans. $\left\{ \begin{array}{l} (a) 11,236. \\ (b) 33,169.515625. \\ (c) .000025. \\ (d) .00003969. \\ (e) 101.2036. \\ (f) 4,603.6225. \\ (g) 936,723,944,025. \\ (h) \frac{1}{256}. \\ (i) \frac{1}{16}. \end{array} \right.$

(111) Solve the following:

(a) 753^2 ; (b) 987.4^2 ; (c) $.005^2$; (d) $.4044^2$; (e) $.0133^2$;
(f) 301.011^2 ; (g) $(\frac{1}{8})^2$; (h) $(3\frac{1}{4})^2$.

Ans. $\left\{ \begin{array}{l} (a) 426,957,777. \\ (b) 962,674,279.624. \\ (c) .000000125. \\ (d) .066135317184. \\ (e) .000002352637. \\ (f) 27,273,890.942264331. \\ (g) \frac{1}{64}. \\ (h) 52.734375. \end{array} \right.$

(112) What is the fifth power of 2? Ans. 32.

(113) What is the fourth power of 3? Ans. 81.

(114) What is the ninth power of 7? Ans. 40,353,607.

(115) Solve the following:

(a) 1.2^4 ; (b) 11^5 ; (c) 1^6 ; (d) $.01^4$; (e) $.1^2$.

Ans. $\left\{ \begin{array}{l} (a) 2.0736. \\ (b) 161,051. \\ (c) 1. \\ (d) .00000001. \\ (e) .00001. \end{array} \right.$

(116) In what respect does evolution differ from involution?

(117) Find the square root of the following:

(a) 3,486,784.401; (b) 9,000,099.4009; (c) .001225;
(d) 10,795.21; (e) 73,008.05; (f) 9; (g) .9.

$$\text{Ans. } \left\{ \begin{array}{l} (a) \ 1,867.29+. \\ (b) \ 3,000.016. \\ (c) \ .035. \\ (d) \ 103.9. \\ (e) \ 270.2. \\ (f) \ 3. \\ (g) \ .948+. \end{array} \right.$$

(118) What is the cube root of the following:

(a) .32768? (b) 74,088? (c) 92,416? (d) .373248?
(e) 1,758.416743? (f) 1,191,016? (g) $\frac{4}{32}$? (h) $\frac{97}{1516}$?

$$\text{Ans. } \left\{ \begin{array}{l} (a) \ .689+. \\ (b) \ 42. \\ (c) \ 45.2115+. \\ (d) \ .72. \\ (e) \ 12.07. \\ (f) \ 106. \\ (g) \ .5. \\ (h) \ .472+. \end{array} \right.$$

(119) Find the cube root of 2 to four decimal places.

Ans. 1.2599 +.

(120) Find the cube root of 3 to three decimal places.

Ans. 1.442 +.

(121) Solve the following:

$$(a) \sqrt[4]{123.21};$$

$$(b) \sqrt[4]{114.921};$$

$$(c) \sqrt[4]{502,681};$$

$$(d) \sqrt[4]{.00041209}.$$

$$\text{Ans. } \left\{ \begin{array}{l} (a) \ 11.1. \\ (b) \ 10.72+. \\ (c) \ 709. \\ (d) \ .0203. \end{array} \right.$$

(122) Solve the following:

$$(a) \sqrt[3]{.0065}; \quad (b) \sqrt[3]{.021}; \quad (c) \sqrt[3]{8,036,054,027};$$

$$(d) \sqrt[3]{.000004096}; \quad (e) \sqrt[3]{17}.$$

$$\text{Ans. } \left\{ \begin{array}{l} (a) .186 +. \\ (b) .27 +. \\ (c) 2,003. \\ (d) .016. \\ (e) 2.57 +. \end{array} \right.$$

(123) Extract the square root of:

$$(a) \frac{1}{2}\frac{2}{3}\frac{4}{5}.$$

$$(b) .3364.$$

$$(c) .1.$$

$$(d) 25.0\frac{3}{4}.$$

$$(e) .000\frac{1}{8}.$$

$$\text{Ans. } \left\{ \begin{array}{l} (a) \frac{3}{4}\frac{1}{2}. \\ (b) .58. \\ (c) .31622 +. \\ (d) 5.00749. \\ (e) .02108. \end{array} \right.$$

(124) Find the value of x in the following:

$$11.7 : 13 :: 20 : x. \quad \text{Ans. } 22.22 +.$$

$$(125) \quad (a) 20 + 7 : 10 + 8 :: 3 : x. \quad \text{Ans. } \left\{ \begin{array}{l} (a) 2. \\ (b) 277.7 +. \end{array} \right.$$

$$(b) 12^2 : 100^2 :: 4 : x.$$

$$(126) \quad (a) \frac{4}{x} = \frac{7}{21}; \quad (b) \frac{x}{24} = \frac{8}{16}; \quad (c) \frac{2}{10} = \frac{x}{100}; \quad (d) \frac{15}{45} = \frac{60}{x};$$

$$(e) \frac{10}{150} = \frac{x}{600}.$$

$$\text{Ans. } \left\{ \begin{array}{l} (a) x = 12. \\ (b) x = 12. \\ (c) x = 20. \\ (d) x = 180. \\ (e) x = 40. \end{array} \right.$$

$$(127) \quad x : 5 :: 27 : 12.5. \quad \text{Ans. } 10\frac{1}{2}.$$

$$(128) \quad 45 : 60 :: x : 24. \quad \text{Ans. } 18.$$

$$(129) \quad x : 35 :: 4 : 7. \quad \text{Ans. } 20.$$

$$(130) \quad 9 : x :: 6 : 24. \quad \text{Ans. } 36.$$

$$(131) \quad \sqrt[3]{1,000} : \sqrt[3]{1,331} :: 27 : x. \quad \text{Ans. } 29.7.$$

$$(132) \quad 64 : 81 :: 21^2 : x^2. \quad \text{Ans. } 23.625.$$

$$(133) \quad 7 + 8 : 7 :: 30 : x. \quad \text{Ans. } 14.$$

(134) If a piece of 2-inch shafting $3\frac{1}{2}$ ft. long weighs 37.45 lb., how much would a piece $6\frac{3}{4}$ ft. long weigh?

$$\text{Ans. } 72.225 \text{ lb.}$$

(135) The intensity of heat from a burning body varies inversely as the square of the distance from it. If a thermometer held 6 ft. from a stove rises 24 degrees, how many degrees will it rise if held 12 ft. from the stove? Ans. 6° .

(136) If sound travels at the rate of 6,160 ft. in $5\frac{1}{2}$ sec., how far does it travel in 1 min.? Ans. 67,200 ft.

(137) If a railway train runs 444 mi. in 8 hr. 40 min., in what time can it run 1,060 mi. at the same rate of speed? Ans. 20 hr. 41.44 min.

(138) If a pump discharging 135 gal. per minute fills a tank in 38 minutes, how long would it take a pump discharging 85 gal. per minute to fill it? Ans. $60\frac{8}{17}$ min.

(139) If a quantity of babbitt metal contains 8 lb. of copper, 8 lb. of antimony, and 80 lb. of tin, how much copper will 32 lb. of the same metal contain? Ans. $2\frac{2}{3}$ lb.

(140) The distances around the drive wheels of two locomotives are 12.56 ft. and 15.7 ft., respectively. How many times will the larger turn while the smaller turns 520 times? Ans. 416 times.

(141) If a cistern 28 ft. long, 12 ft. wide, 10 ft. deep holds 510 bbl. of water, how many barrels of water will a cistern hold that is 20 ft. long, 17 ft. wide, and 6 ft. deep? Ans. $309\frac{3}{4}$ bbl.

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MENSURATION AND USE OF LETTERS

IN

ALGEBRAIC FORMULAS.

(SEE ARTS. 340-432.)

$$\begin{array}{ll} A = 5 & h = 200 \\ B = 10 & x = 12 \\ i = 3.5 & D = 120 \end{array}$$

Work out the solutions to the following formulas, using the above values for the letters:

$$(142) \quad C = \frac{D - x}{B + i}. \quad \text{Ans. } C = 8.$$

$$(143) \quad Q = \frac{A h + D}{2x + 6} + D. \quad \text{Ans. } Q = 157\frac{1}{3}.$$

$$(144) \quad r = \frac{3.246 B h}{\frac{A x + h}{A i - B}}. \quad \text{Ans. } r = 187.269 +.$$

$$(145) \quad v = \sqrt{\frac{A D}{i B + 1.5}}. \quad \text{Ans. } v = 4.05 +.$$

$$(146) \quad u = \sqrt[3]{\frac{B x}{.00018 h (A^2 - x)}}. \quad \text{Ans. } u = 6.35 +.$$

$$(147) \quad f = \frac{10 (h - D)^2}{\sqrt[3]{D + A}}. \quad \text{Ans. } f = 12,800.$$

$$(148) \quad g = \frac{(B - A)^2 - \sqrt[3]{D + A}}{A^2 - (1 + D)}. \quad \text{Ans. } g = 5.$$

$$(149) \quad k = \sqrt{\frac{A B^2}{\sqrt[3]{A h}}}. \quad \text{Ans. } k = 7.071 +.$$

$$(150) \quad T = \sqrt{\frac{A^2 \left[490 + \frac{(hx)^2}{D^2} \right]}{h + \frac{x}{D} (A^2 - B)^2}}. \quad \text{Ans. } T = 10.$$

(151) If one of the angles formed by one straight line meeting another straight line equals $152^\circ 3'$, what is the other angle equal to? Ans. $27^\circ 57'$.

(152) How many seconds are there in $140^\circ 17' 10''$?
Ans. 505,030".

(153) Write a definition for a degree, as applied to the measurement of angles.

(154) (a) How many degrees are there in 240 minutes?
 (b) How many seconds? Ans. $\begin{cases} (a) 4^\circ. \\ (b) 14,400". \end{cases}$

(155) Draw an obtuse angle, a right angle, and an acute angle. State the name of each angle by using letters to designate them.

(156) Draw a rhombus and then draw a rectangle having the same area.

(157) Can a quadrilateral be formed with lines whose lengths are 20 inches, 9 inches, 4 inches, and 7 inches? Give reasons.

(158) A sheet of zinc measures $11\frac{1}{2}$ inches by $2\frac{1}{2}$ feet. How many square inches does it contain? Ans. 345 sq. in.

(159) If the zinc in the last example weighs $5\frac{1}{4}$ pounds, what is its weight per square foot? Ans. 2.19 lb.

(160) How many boards 16 feet long and 5 inches wide would be required to lay a floor measuring 15×24 feet?
Ans. 54 boards.

(161) A lot of land is in the shape of a trapezoid. It is 16 rods long, 9 rods wide at the front, and 6 rods wide at the back. The front and back being parallel, what part of an acre does the lot contain? Ans. $\frac{3}{4}$ of an acre.

- (162) The accompanying figure shows the floor plan of

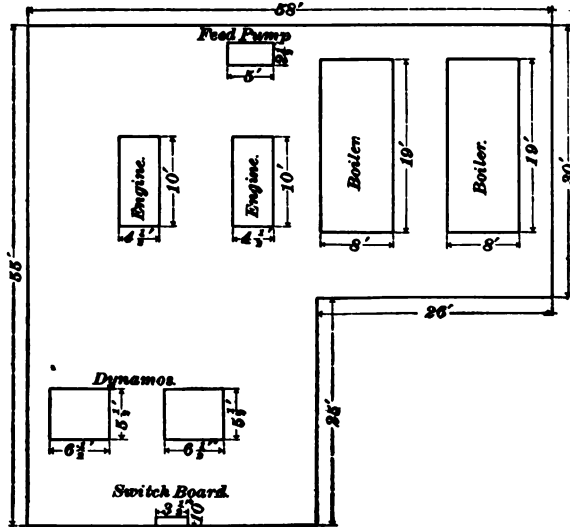


FIG. 1.

an electric-light station. From the dimensions given, calculate the number of square feet of unoccupied floor space.

Ans. 2,059.08 sq. ft.

- (163) A sidewalk 10 feet wide extends around a city block 528 feet long and 352 feet wide. Assuming that there is no space between the walk and the buildings, how many square yards does the walk contain?

Ans. 2,000 sq. yd.

- (164) How many square yards of plastering will be required for the four side walls of a hall 90 feet long, 50 feet wide, and 20 feet high, with four doors $5\frac{1}{2} \times 10$ feet and fourteen windows 5×11 feet? There is a baseboard 9 inches high.

Ans. 490.72 sq. yd.

- (165) A triangle has three equal angles ; what is it called ?

- (166) If a triangle has two equal angles, what kind of a triangle is it ?

- (167) Can a triangle be formed with three lines whose lengths are 12 inches, 7 inches, and 4 inches ? Give reasons for your opinion.

(168) (a) What is the altitude of an equilateral triangle whose sides are each 6 feet? (b) What is its area?

Ans. $\left\{ \begin{array}{l} (a) \text{ 5.196 ft.} \\ (b) \text{ 15.588 sq. ft.} \end{array} \right.$

(169) In a triangle ABC , angle $A = 23^\circ$, and $B = 32^\circ 32'$; what does angle C equal?

Ans. $C = 124^\circ 28'$.

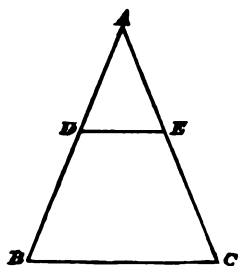


FIG. 2.

(170) In the figure, if $AD = 10$ inches, $AB = 24$ inches, and $BC = 13\frac{1}{2}$ inches, how long is DE , DE being parallel to BC ?

Ans. $DE = 5.625$ in.

(171) An engine room is 52 feet long and 39 feet wide. How many feet is it from one corner to a diagonally opposite one, measured in a straight line?

Ans. 65 ft.

(172) A ladder 24 feet long rests against a house with its upper end 8 feet from the ground. How far on the ground is the lower end of the ladder from the house?

Ans. $22 \text{ ft. } 7\frac{1}{2} + \text{in.}$

(173) (a) Show why it is that the area of a triangle equals one-half the product of the base by the altitude. (b) Does it make any difference which side is taken as the base?

(174) (a) The area of an isosceles triangle is 200 square inches. If its altitude is 20 inches, how long is its base? (b) What is the length of one of its equal sides?

Ans. $\left\{ \begin{array}{l} (a) \text{ 20 in.} \\ (b) \text{ 22.36 in.} \end{array} \right.$

(175) In an equilateral heptagon one of the sides equals 3 inches; what is the length of the perimeter?

Ans. 21 in.

(176) The perimeter of a regular decagon is 40 inches; what is the length of the side?

Ans. 4 in.

(177) What is one angle of a regular dodecagon equal to?

Ans. 150° .

(178) The area of a regular pentagon is 43 square inches. If one side is 5 inches long, what is the perpendicular distance from the center to one side?

Ans. 3.44 in.

(179) It is required to make a miter-box, in which to cut molding to fit around an octagon post. At what angle with the side of the box should the saw run? Ans. $67\frac{1}{2}^\circ$.

(180) Calculate the area of the irregular polygon, Fig. 3. The dimensions are to be obtained by measuring. Ans. $1.78 + \text{sq. in.}$

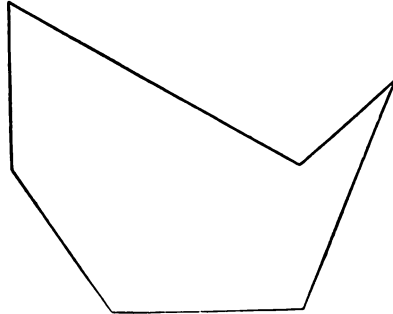


FIG. 3.

(182) If the distance between two opposite corners of a hexagonal nut is two inches, what is the distance between two opposite sides? Ans. $1.732 + \text{in.}$

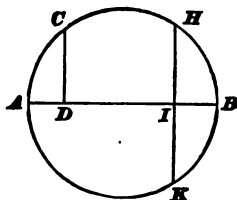


FIG. 4.

(183) In the accompanying figure, if the distance BI is 6 inches and HK 18 inches, what is the diameter of the circle? Ans. 19.5 in.

(184) In the same figure, if the diameter $AB = 32\frac{1}{2}$ feet, and the distance $IB = 8$ feet, what is the length of the chord HK ? Ans. 28 ft.

(185) The trunk of a tree measures 7.854 feet around it; what is its diameter? Ans. $2\frac{1}{2}$ ft.

(186) How many revolutions will a 72-inch locomotive driver make in going one mile? Ans. 280.112 revolutions.

(187) A pipe has an internal diameter of 6.06 inches; what is the area of a circle having this diameter? Ans. 28.8427 sq. in.

(188) The area of a circle having a diameter corresponding to the internal diameter of a certain pipe is 113.0973 square inches. What is the outside diameter, the pipe being $\frac{3}{4}$ inch thick? Ans. $13\frac{1}{2}$ inches.

(189) How long must the arc of a circle be to contain 12° , supposing the radius of the circle to be 6 inches?

Ans. 1.25664 in.

(190) Calculate the area of a flat circular ring whose outside diameter is 22 inches, and whose inside diameter is 21 inches.

Ans. 33.7722 sq. in.

(191) Find the area of a segment of a circle whose diameter is 56 inches, the height of the segment being 5 inches.

Ans. 108.5 sq. in.

(192) What are the dimensions of the end of the largest square bar that can be planed from an iron bar 2 inches in diameter?

Ans. 1.4142 in. square.

(193) What is the area of the sector of a circle 15 inches in diameter, the angle between the two radii forming the sector being $12\frac{1}{2}^\circ$?

Ans. 6.1359 sq. in.

(194) (a) What would be the length of the side of a square metal plate having an area of 103.8691 square inches? (b) What would be the diameter of a round plate having this area? (c) How much shorter is the circumference of the round plate than the perimeter of the square plate?

Ans. $\left\{ \begin{array}{l} (a) \text{ 10.1916 in.} \\ (b) \text{ } 11\frac{1}{2} \text{ in.} \\ (c) \text{ 4.638 in.} \end{array} \right.$

(195) A plate 46 inches long is to be rolled into a shell; what will be the diameter of the shell if 5 inches are allowed for lap?

Ans. 13.05 in.

(196) What is the convex area in square feet of a section of a smokestack 26 inches in diameter and $10\frac{1}{2}$ feet long?

Ans. 71.4714 sq. ft.

(197) Find the area in square feet of the entire surface of a hexagonal column 12 feet long, each edge of the ends of the column being 4 inches long.

Ans. 24.5774 sq. ft.

(198) Find the cubical contents of the above column in cubic inches.

Ans. 5,985.9648 cu. in.

(199) Compute the weight per foot of an iron boiler tube 4 inches outside diameter and 3.73 inches inside diameter, the weight of the iron being taken at .28 pound per cubic inch.

Ans. $5\frac{1}{2}$ lb.

(200) The dimensions of a return-tubular boiler are as follows: Diameter, 60 inches; length between heads, 16 feet; outside diameter of tubes, $3\frac{1}{2}$ inches; number of tubes, 64; distance of mean water-line from top of boiler, 18 inches.

(a) Compute the steam space of the boiler in cubic feet.

(b) Determine the number of gallons of water that will be required to fill the boiler up to the mean water level.

Ans. $\left\{ \begin{array}{l} (a) \text{ 79.2 cu. ft.} \\ (b) \text{ 1,246 gal., nearly.} \end{array} \right.$

(201) The cylinders of a compound engine are 19 and 31 inches in diameter, and the stroke is 24 inches; if the clearance at *each* end of the small cylinder is 14% of the *stroke*, and in the large cylinder 8% of the *stroke*, (a) what is the total volume in cubic feet of the steam in the small cylinder during one stroke? (b) In the large cylinder? (c) What is the ratio between the two?

Ans. $\left\{ \begin{array}{l} (a) \text{ 4.489 cu. ft.} \\ (b) \text{ 11.321 cu. ft.} \\ (c) \text{ Ratio} = 2.522 : 1. \end{array} \right.$

(202) Find the volume of a triangular pyramid 10 inches high; the base is an equilateral triangle, and one edge measures 10 inches.

Ans. 144.336 cu. in.

(203) The slant height of a square pyramid is 25 inches, and one edge of its base is 16 inches. Find its altitude by the principle of the right-angled triangle.

Ans. 23.6854 in.

(204) The length of the circumference of the base of a cone is 18.8496 inches, and its slant height is 10 inches. Find the area of the entire surface of the cone.

Ans. 122.5224 sq. in.

(205) If the altitude of the above cone were 9 inches, what would be its volume?

Ans. 84.8232 cu. in.

(206) A square vat is 11 feet deep, 15 feet square at the top, and 12 feet square at the bottom. How many gallons will it hold?

Ans. 15,058.29 gal.

(207) How many pails of water would be required to fill the vat, the pail having the following dimensions: Depth, 11 inches; diameter at the top, 12 inches; diameter at the bottom, 9 inches? Ans. 3,627.28.

(208) Required the area of the convex surface of the frustum of a hexagonal pyramid whose slant height is 32 inches, the perimeter of the lower base measuring 48 inches, and of the upper base 36 inches. Ans. 1,344 sq. in.

(209) It is desired to find the number of gallons that a certain tank will hold. By measuring with a tape measure, it is found to be 190 inches in circumference at the bottom and $170\frac{1}{2}$ inches in circumference at the top. The depth of the tank is 7 feet, and the thickness of the sides $1\frac{1}{4}$ inches. Make the calculation in round numbers. Ans. 861 gal.

NOTE.—To obtain the area of the upper and lower bases, first find the outside diameters and then deduct the thickness of the walls, or $2 \times 1\frac{1}{4}$ inches.

(210) Find (*a*) the area of the surface, and (*b*) the cubical contents of a ball $22\frac{1}{2}$ inches in diameter.

Ans. $\left\{ \begin{array}{l} (a) \text{ 1,590.435 sq. in.} \\ (b) \text{ 5,964.1313 cu. in.} \end{array} \right.$

(211) What is the volume of a ball whose surface has an area of 201.0624 square inches? Ans. 268.0832 cu. in.

(212) (*a*) What is the volume and area of a cylindrical ring whose outside diameter is 16 inches and inside diameter 13 inches? (*b*) If made of cast iron, what is its weight? Take the weight of 1 cubic inch of cast iron as .261 pound.

Ans. Weight = 21 lb.

(213) The volume of a certain cylindrical ring is 144.349 cu. in., its length before bending (see line *D* in Fig. 61, Art. 432) was 20.42 inches; what is the area of its surface?

Ans. 192.45 sq. in.

ELEMENTARY ALGEBRA

AND

TRIGONOMETRIC FUNCTIONS.

(ARTS. 433-617.)

(214) How may the signs of all the terms of the denominator of a fraction be changed from + to - or from - to + without altering the value of the fraction?

(215) Solve the equations:

(a) $3x + 6 - 2x = 7x$; (b) $5x - (3x - 7) = 4x - (6x - 35)$;
 (c) $(x + 5)^2 - (4 - x)^2 = 21x$.

Ans. $\begin{cases} (a) & x = 1. \\ (b) & x = 7. \\ (c) & x = 3. \end{cases}$

(216) Divide $3a^3 + 2 - 4a^5 + 7a + 2a^6 - 5a^4 + 10a^2$ by $a^3 - 1 - a^2 - 2a$.

Ans. $2a^3 - 2a^2 - 3a - 2$.

(217) Multiply: (a) $2 + 4a - 5a^2 - 6a^3$ by $7a^3$; (b) $4x^3 - 4y^3 + 6z^3$ by $3x^2y$; (c) $3b + 5c - 2d$ by $6a$.

Ans. $\begin{cases} (a) & 14a^3 + 28a^4 - 35a^5 - 42a^6. \\ (b) & 12x^5y - 12x^2y^3 + 18x^2yz^3. \\ (c) & 18ab + 30ac - 12ad. \end{cases}$

(218) A man performed a journey of 48 miles in a certain number of hours; but if he had traveled 4 miles more each hour, he would have performed the journey in 6 hours. How many miles did he travel per hour? Ans. 4 miles.

(219) Translate the following algebraic expressions into ordinary language: $\sqrt[n]{\frac{a+b+c}{n}} + \sqrt[n]{a} + \frac{b+c}{n} + \sqrt[n]{a+b} +$

$\frac{c}{n} + (a+b)c + a + bc$.

(220) Find the products of:

$$(a) \frac{9m^3n^3}{8p^3q^3}, \frac{5p^3q}{2xy}, \text{ and } \frac{24x^2y^3}{90mn}; \quad \text{Ans. } \begin{cases} (a) \frac{3mnxy}{4pq^3} \\ (b) \frac{a}{3ax+4} \end{cases}$$

$$(b) 3ax + 4 \text{ and } \frac{a^3}{9a^3x^3 + 24a^2x + 16a}.$$

(221) Find the sine, cosine, and tangent of $17^\circ 27' 37''$.

(222) A vessel containing some water was filled by pouring in 42 more gallons; there was then seven times as much water in the vessel as at first. How much did the vessel hold?
Ans. 49 gallons.

(223) Reduce $\frac{c(a+b)+cd}{(a+b)c}$ to its simplest form.

(224) Find the sine, cosine, and tangent of $63^\circ 4' 51.8''$.

(225) Find the values of the following:

$$\left(c^{-\frac{1}{2}}\right)^{-\frac{1}{2}}; \left(m\sqrt{n^3}\right)^{-\frac{1}{2}}; \left(cd^{-3}\right)^{\frac{1}{a}}.$$

(226) Simplify $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}$. Ans. $\frac{1}{x+2}$.

(227) Multiply $\frac{a^2+c^2+ac}{a^2+b^2-c^2-2ab}$ by $\frac{a^2+c^2-b^2-2ac}{a^3c+a^2c^2+ac^3}$.

SUGGESTION.—Factor the numerators and denominators before multiplying.

$$\text{Ans. } \frac{a+b-c}{ac(a-b+c)}.$$

(228) From $a^4 - b^4$ take $5a^3b - 7a^2b^2 + 5ab^3$, and from the result take $3a^4 - 4a^3b + 6a^2b^2 + 5ab^3 - 3b^4$.

$$\text{Ans. } -2a^4 - a^3b + a^2b^2 - 10ab^3 + 2b^4.$$

(229) Sine = .27038, cosine = .27038, and tangent = 2.27038; find the corresponding angles.

(230) (a) Give an illustration, not contained in the section on Algebra, that will explain the difference between positive and negative quantities. (b) In what respects are addition and subtraction different in algebra from addition and subtraction in arithmetic?

(231) (a) What is the value of a^0 ? (b) What does $a^0 \div a^{-1}$ equal?

(232) Change the fraction $-\frac{c-(a-b)}{c+(a+b)}$ so that the sign before the dividing line will be +.

(233) (a) What is the reciprocal of $\frac{2}{3}$? (b) Of what number is 700 the reciprocal?

(234) (a) From $3a - 2b + 3c$ take $2a - 7b - c - b$. (b) Subtract $x^3 + y^3 - xy^2$ from $2x^3 - 3x^2y + 2xy^2$. (c) From $14a + 4b - 6c - 3d$ take $11a - 2b + 4c - 4d$.

$$\text{Ans. } \begin{cases} (a) & a + 6b + 4c. \\ (b) & x^3 - 3x^2y + xy^2 + 2xy^2 - y^3. \\ (c) & 3a + 6b - 10c + d. \end{cases}$$

(235) Find the numerical values of the following expressions when $a = 16$, $b = 10$, and $x = 5$: (a) $(ab^2x + 2abx)4a$; (b) $2\sqrt{4a} - \frac{2bx}{a-b} + \frac{b-x}{x}$; (c) $(b - \sqrt{a})(x^3 - b^2)(a^2 - b^2)$.

$$\text{Ans. } \begin{cases} (a) & 614,400. \\ (b) & \frac{1}{3}. \\ (c) & 23,400. \end{cases}$$

(236) Find the sum of the following: (a) $4xyz - 5xyz$, $6xyz - 9xyz + 3xyz$; (b) $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, $-a^2 + 5ab - b^2$, $18a^2 - 20ab - 19b^2$, and $14a^2 - 3ab + 20b^2$; (c) $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4p$.

$$\text{Ans. } \begin{cases} (a) & -4xyz. \\ (b) & 39a^2 - 24ab + 5b^2. \\ (c) & 6mn - ab - 4c + 3x + 3m^2 - 4p. \end{cases}$$

(237) (a) Explain in your own words the difference between a coefficient and an exponent. (b) How are coefficients and exponents treated in multiplication, and how in division? (c) What is the law of signs in multiplication?

(238) In a right triangle ABC , the hypotenuse $AB = 17.69$ feet, and the side $AC = 9$ ft. 9 in.; find the other three parts.

$$\text{Ans. } \begin{cases} 56^\circ 33' 15''. \\ 33^\circ 26' 45''. \\ 14 \text{ ft. } 9 \text{ in.} \end{cases}$$

(239) Solve the equations:

$$(a) \frac{9x+20}{36} = \frac{4(x-3)}{5x-4} + \frac{x}{4}; \quad (b) ax - \frac{3a-bx}{2} = \frac{1}{2};$$

$$(c) am - b - \frac{ax}{b} + \frac{x}{m} = 0.$$

$$\text{Ans. } \begin{cases} (a) x = 8. \\ (b) x = \frac{3a+1}{2a+b}. \\ (c) x = bm. \end{cases}$$

(240) State how you would read the following expressions:

$$(a) a^2x^2 + 2a^2b^2 - (a+b); \quad (b) \sqrt[3]{x} + y(a-n^2)^{\frac{1}{2}};$$

$$(c) (m+n)(m-n)^2 \left(m - \frac{n}{2}\right).$$

(241) (a) Write a monomial; a binomial; a polynomial.
 (b) In the expression $a + 2ab - b^2$ why can not the indicated addition and subtraction be performed? (c) What operation is indicated between the quantities in $4ac^2d$?

(242) Resolve into their factors: (a) $45x^3y^{10} - 90x^2y^7 - 360x^4y^2$; (b) $a^2b^2 + 2abcd + c^2d^2$; (c) $(a+b)^2 - (c-d)^2$.

$$\text{Ans. } (c) (a+b+c-d)(a+b-c+d).$$

(243) Remove the symbols of aggregation from the following:

$$(a) 2a - \{3b + [4c - 4a - (2a + 2b)] + [3a - \overline{b+c}]\};$$

$$(b) 7a - \{3a - [(2a - 5a) + 4a]\};$$

$$(c) a - \{2b + [3c - 3a - (a + b)] + [2a - (b + c)]\}.$$

$$\text{Ans. } \begin{cases} (a) 5a - 3c. \\ (b) 5a. \\ (c) 3a - 2c. \end{cases}$$

(244) In a right triangle ABC , right-angled at C , side $AC = 17.5$, side $BC = 21.3$; find the other three parts.

$$\text{Ans. } \begin{cases} 39^\circ 24' 23''. \\ 50^\circ 35' 37''. \\ 27.57 \text{ nearly.} \end{cases}$$

(245) The following formula appears in the section on Heat: $t = \frac{W_1s_1t_1 + W_2s_2t_2}{W_1s_1 + W_2s_2}$. It is required to transform it so

that t_1 will stand alone in the first member. In other words, solve for t_1 .

$$\text{Ans. } t_1 = \frac{(W_1 s_1 + W_2 s_2)t - W_1 s_1 t_1}{W_2 s_2}$$

(246) A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, working together? Ans. 2 days.

$$(247) \text{ Simplify: } (a) \frac{\frac{1}{1-x} - \frac{1}{1+x}}{\frac{1}{1-x} + \frac{1}{1+x}}; \quad (b) \frac{\frac{a^2}{b^2} + \frac{1}{a}}{\frac{a}{b^2} - \frac{a-b}{ab}};$$

$$(c) \frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}.$$

$$\text{Ans. } \begin{cases} (a) x. \\ (b) \frac{a+b}{b}. \\ (c) \frac{4}{3x+3}. \end{cases}$$

(248) (a) Arrange $a^2b^2 + 2abc + 3 - 7a^2b^2 + 6a^2b^2$ according to the decreasing powers of a ; (b) according to the increasing powers of b . (c) With $a^2 + 1 + 2a^2 + ax$ arranged according to the *increasing* powers of a , should the 1 be placed first or last, and why?

(249) (a) Express with radical signs: $x^{\frac{1}{2}}$; $3x^{\frac{1}{2}}y^{-\frac{1}{2}}$; $3x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{2}}$. (b) Clear $a^{-1}b^{\frac{1}{2}} + \frac{c^{-2}}{a+b} + (m-n)^{-1} - \frac{a^2b^{-2}c}{c^{-3}}$ of negative exponents. (c) Express with fractional exponents: $\sqrt[7]{x^6}$; $\sqrt[3]{x^{-1}}$; $(\sqrt[4]{b^5x^2})^3$.

(250) Divide:

$$(a) \frac{2ax + x^2}{a^2 - x^2} \text{ by } \frac{x}{a-x}; \quad (b) \frac{6m^2n^2 - 3n}{4m^4n^2 - 4m^2n + 1} \text{ by } \frac{3n}{4m^4n^2 - 1};$$

$$(c) 9 + \frac{5y^2}{x^2 - y^2} \text{ by } 3 + \frac{5y}{x-y}.$$

$$\text{Ans. } \begin{cases} (a) \frac{2a+x}{a^2+ax+x^2}. \\ (b) 2m^2n+1. \\ (c) \frac{3x-2y}{x+y}. \end{cases}$$

(251) Multiply: (a) $2x^3 + 2x^2 + 2x - 2$ by $x - 1$; (b) $x^3 - 4ax + c$ by $2x + a$; and (c) $-a^3 + 3a^2b - 2b^3$ by $5a^2 + 9ab$.

$$\text{Ans. } \begin{cases} (a) 2x^4 - 4x + 2. \\ (b) 2x^3 - 7ax^2 + 2cx - 4a^2x + ac. \\ (c) -5a^5 + 6a^4b + 27a^3b^2 - 10a^2b^3 - 18ab^4. \end{cases}$$

(252) Perform the indicated additions:

$$(a) \frac{x}{x-y} + \frac{x-y}{y-x}; \quad (b) \frac{x^2}{x^2-1} + \frac{x}{x+1} - \frac{x}{1-x};$$

$$(c) \frac{3a-4b}{7} - \frac{2a-b+c}{3} + \frac{13a-4c}{12}.$$

$$\text{Ans. } \begin{cases} (a) \frac{y}{x-y}. \\ (b) \frac{3x^2}{x^2-1}. \\ (c) \frac{71a-20b-56c}{84}. \end{cases}$$

(253) Factor the following: (a) $9x^4 + 12x^2y^2 + 4y^4$; (b) $49a^4 - 154a^2b^2 + 121b^4$; (c) $64x^2y^3 + 64xy + 16$.

(254) Divide: (a) $3x^3 + x + 9x^2 - 1$ by $3x - 1$; (b) $a^3 - 2ab^2 + b^3$ by $a - b$; (c) $7x^3 + 58x - 24x^2 - 21$ by $7x - 3$.

$$\text{Ans. } \begin{cases} (a) 3x^2 + 2x + 1. \\ (b) a^2 + ab - b^2. \\ (c) x^2 - 3x + 7. \end{cases}$$

(255) Why are letters used in algebra, and in what ways do they differ from figures?

(256) (a) Reduce $1 + 2x - \frac{4x-4}{5x}$ to a fractional form.

(b) Change $\frac{3x^2+2x+1}{x+4}$ to a mixed quantity.

$$\text{Ans. } \begin{cases} (a) \frac{10x^2+x+4}{5x}. \\ (b) 3x - 10 + \frac{41}{x+4}. \end{cases}$$

(257) In a right triangle ABC , angle $A = 65^\circ 13' 29''$, hypotenuse $AB = 5\frac{1}{2}$ yards; find the other three parts. Give sides in feet and inches.

$$\text{Ans. } \begin{cases} 24^\circ 46' 31'' \\ 14 \text{ ft. } 11\frac{3}{4} \text{ in.} \\ 6 \text{ ft. } 11 \text{ in., nearly.} \end{cases}$$

(258) What is the ratio of (a) $x^4 - 1$ to $x^2 + 1$? (b) $x^4 - 2x^2y^2 + y^4$ to $x - y$?

$$\text{Ans. (b) } (x^2 - y^2)(x + y).$$

(259) Solve the following:

$$(a) \frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1);$$

$$(b) (a^2 + x)^2 = x^2 + 4a^2 + a^4; (c) \frac{x-1}{x-2} - \frac{x+1}{x+2} = \frac{3}{x^2-4}.$$

$$\text{Ans. } \begin{cases} (a) x = 1\frac{1}{2} \\ (b) x = 2 \\ (c) x = 1\frac{1}{2} \end{cases}$$

(260) Add $159^\circ 27' 34.6''$, $25^\circ 16' 8.7''$, and $3^\circ 48' 53''$.

(261) Combine the like terms of the expressions: (a) $x + y + z - (x - y) - (y + z) - (-y)$; (b) $(2x - y + 4z) + (-x - y - 4z) - (3x - 2y - z)$; (c) $a - [2a + (3a - 4a)] - 5a - \{6a - [(7a + 8a) - 9a]\}$.

$$\text{Ans. } \begin{cases} (a) 2y. \\ (b) z - 2x. \\ (c) -5a. \end{cases}$$

(262) Divide: (a) $35m^2y + 28m^2y^2 - 14my^3$ by $-7my$; (b) $4a^4 - 3a^3b - a^2b^2$ by a^2 ; (c) $4x^5 - 8x^3 + 12x^2 - 16x^2$ by $4x^2$.

$$\text{Ans. } \begin{cases} (a) -5m^2 - 4my + 2y^2. \\ (b) 4 - 3ab - a^2b^2. \\ (c) x - 2x^3 + 3x^5 - 4x^7. \end{cases}$$

(263) A post has $\frac{1}{4}$ of its length in the earth, $\frac{3}{4}$ in the water, and 13 feet in the air. What is its length?

$$\text{Ans. } 35 \text{ feet.}$$

(264) In the composition of a quantity of gunpowder, the niter was 10 lb. more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ lb. less than $\frac{1}{3}$ of the whole, and the charcoal was

334 EL. ALGEBRA AND TRIG. FUNCTIONS.

2 lb. less than $\frac{1}{2}$ of the mixer. What was the amount of gunpowder and of each of the ingredients?

Ans. $\left\{ \begin{array}{l} \text{Gunpowder, 69 lb.} \\ \text{Mixer, 36 lb.} \\ \text{Sulphur, 7 lb.} \\ \text{Charcoal, 6 lb.} \end{array} \right.$

(265) Find the values of the following:

(a) $\sqrt[3]{-125x^3y^3z^3}$; (b) $\sqrt[3]{10,000a^3b^3c^3}$; (c) $\sqrt[3]{243m^3n^3}$.

(d) $\sqrt[3]{-\frac{x^3y^3z^3}{a^3b^3c^3d^3}}$

Ans. $\left\{ \begin{array}{l} (a) -5xy^2z^2. \\ (b) \pm 10a^4b^3c^2. \\ (c) 3m^2n^2. \\ (d) -\frac{xyz^2}{a^2b^2c^2d} \end{array} \right.$

(266) In a triangle ABC , side $AB = 70$ feet, side $BC = 42$ feet, and angle $A = 36^\circ 10'$. Find the angles B and C and the side AC .

Ans. $\left\{ \begin{array}{l} \text{Angle } B = 64^\circ 14'. \\ \text{Angle } C = 79^\circ 36'. \\ \text{Side } AC = 64.1 \text{ ft.} \end{array} \right.$

(267) (a) What is the supplement of $72^\circ 11' 36''$? (b) What is the complement of $22^\circ 34' 17''$?

Ans. $\left\{ \begin{array}{l} (a) 107^\circ 48' 24''. \\ (b) 67^\circ 25' 43''. \end{array} \right.$

(268) In a triangle ABC , angle $A = 57^\circ 34.5'$, angle $C = 44^\circ 22.5'$, and side $AB = 344$ feet. Find angle B and the sides AC and BC .

Ans. $\left\{ \begin{array}{l} \text{Angle } B = 78^\circ 3'. \\ \text{Side } AC = 481.22 \text{ ft.} \\ \text{Side } BC = 415.19 \text{ ft.} \end{array} \right.$

LOGARITHMS.

(ARTS. 618-667.)

(269) Solve, using logarithms,

$$x = 351.36 \times 100 \times 24[1 - (\frac{2}{100})^{.39078}].$$

NOTE.—In logarithmic work negative quantities are used as though they were positive, the sign of the result being determined independently.

Ans. $x = -188,300$.

(270) What are the logarithms of the following numbers:

(a) 2,376? (b) .6413? (c) .0002507?

(271) Divide the following by using logarithms:

(a) $755.4 \div .00324$; (b) $.05555 \div .0008601$; (c) $4.62 \div .6448$.

Ans. $\begin{cases} (a) 233,150. \\ (b) 64.584. \\ (c) 7.1648. \end{cases}$

(272) Using logarithms, find the value of x in

$$x^{.74} = \frac{238 \times 1,000}{.0042^{.6602}}.$$

Ans. $x = 2,432,700,000$.

(273) Divide $\sqrt[5]{.00743}$ by $\sqrt[5]{.006}$.

Ans. 1,893.6.

(274) Multiply together the following numbers by using logarithms: 1,728, .00024, .7462, 302.1, and 7.6094.

Ans. 711.40.

(275) Calculate the value of $\frac{\sqrt{5.954} \times \sqrt[3]{61.19}}{\sqrt[5]{298.54}}$.

Ans. 3.0759.

(276) Calculate the value of $\sqrt[7]{.0532864}$. Ans. .65780.

(277) Obtain the values of: (a) $32^{4.8}$; (b) $.76^{3.42}$, and (c) $.84^{.32}$.

Ans. $\begin{cases} (a) 16,777,000. \\ (b) .37028. \\ (c) .93590. \end{cases}$

(278) Calculate the value of $\sqrt[5]{\frac{1}{249}} \div \sqrt[5]{\frac{23}{71}}$.

Ans. .49950.

(279) Find the numbers corresponding to the following logarithms: .81293, 2.52460, $\bar{1}.27631$.

(280) Find the value of v_1 in $p v_1^{.41} = p_1 v_1^{1.41}$, when $p = 134.7$, $v = 1.495$, and $p_1 = 16.421$.

Ans. 6.6504.

(281) What is the value of

$$\sqrt[5]{\frac{7.1895 \times 4,764.2^3 \times 0.00326^3}{.000489 \times 457^3 \times .576^3}}?$$

Ans. .020786.

(282) In the formula $p = 960,000 \frac{t^{2.18}}{ld}$, find the value of p , when $t = \frac{3}{16}$, $l = 120$, and $d = 2\frac{1}{4}$.

Ans. 92.480.

(283) Referring to example 282, what is the value of t , when $p = 160$, $l = 132$, and $d = 2$?

Ans. .23863.

ELEMENTARY MECHANICS.

(ARTS. 828-966.)

(355) A ball, thrown horizontally by the hand, has a velocity of 500 ft. per second. If the ground is level, and the distance from the ground to the hand at the instant the ball leaves the hand is 5 ft. 6 in., how far will the ball go before striking the ground?

Ans. 292.42 ft.

(356) An engine fly-wheel, 80 in. in diameter, makes 160 revolutions per minute; what is the velocity in feet per second of a point on the rim?

Ans. 55.85 ft. per sec.

(357) In the last example, through how many degrees, minutes and seconds will a point on the rim turn in one-seventh of a second?

Ans. $137^{\circ} 8' 34\frac{1}{2}''$.

(358) The fly-wheel of an engine drives a pulley, rigidly connected to a drum on which an elevator rope winds. The fly-wheel is 4 ft. in diameter and makes 54 revolutions per minute; the diameter of the pulley is 36 in., and of the drum, 18 in.; (a) how long will it take the elevator to reach the top of a building 100 ft. high? (b) If required to reach it in 30 seconds, how many revolutions should the fly-wheel make per minute?

Ans. $\left\{ \begin{array}{l} (a) \text{ 17.68 sec.} \\ (b) \text{ 31.83 R. P. M.} \end{array} \right.$

(359) Define and give an example of uniform motion; of variable motion.

(360) Define force. Name five different kinds of forces.

(361) Explain what you understand by inertia.

(362) Is inertia a force? If so, why?

(363) What is weight? How is it measured?

(364) In order that the effect of a force upon a body may be compared with that of another force acting on the same body, what three conditions must be fulfilled?

(365) What is motion? What is rest?

(366) Can a body be in motion with respect to one body and at rest with respect to another? Give examples.

(367) Where will a body weigh the more, on the top of a high mountain or at the bottom of a deep valley, the bottom of the valley being as far below sea-level as the top of the mountain is above?

(368) If the top of a mountain is 31,680 feet above sea-level, what would a body weighing 20,000 lb. at sea-level weigh at the top of the mountain? Take the radius of the earth at sea-level as 3,960 miles. Ans. 19,939 lb. $8\frac{1}{2}$ oz.

(369) If the body in the last example had been dropped in a hole just big enough to let it fall to the bottom, and the bottom of the hole was 2 miles below sea-level, how much would it have weighed at the bottom of the hole?

Ans. 19,989 lb. 14.4 oz.

(370) State the three laws of motion.

(371) What is acceleration?

(372) What do you understand by initial velocity?

(373) Why is it dangerous to jump from a moving train?

(374) Why is it that a man cannot lift himself by pulling his boot straps?

(375) Explain how forces are represented by lines.

(376) What is meant by the expression, "the resultant of several forces"?

(377) If a line 5 in. long represents a force of 20 lb., (a) how long must the line be to represent a force of 1 lb.? (b) Of $6\frac{1}{4}$ lb.?

(378) What do you understand by the components of a force?

(379) If a body be acted upon by two equal forces, one due east and the other due south, in what direction will the body move? What is the direction of the resultant of the two forces?

(380) Find the point of suspension of a rectangular cast iron lever 4 ft. 6 in. long, 2 in. deep and $\frac{3}{4}$ in. thick, having

weights of 47 lb. and 71 lb. hung from each end, in order that there may be equilibrium. Take the weight of a cubic inch of cast iron as .261 lb.

SUGGESTION.—First find the weight of the lever; then consider this weight to be concentrated at the center of gravity of the lever, and combine it with the other two weights in the same manner as though there were three weights.

$$\text{Ans. } \begin{cases} \text{Short arm} = 22.343 \text{ in.} \\ \text{Long arm} = 31.657 \text{ in.} \end{cases}$$

Solve the two following examples by the method of triangle of forces, and parallelogram of forces, and mark the direction of the resultant:

(381) Two forces act upon a body at a common point; one with a force of 75 lb., and the other with a force of 40 lb.; if the angle between them is 60° , and both forces act towards the body, what is the value of the resultant?

Ans. 101.12 lb.

(382) In the last example, if one force (the one of 75 lb.) acts away from the body, and the other towards it, what is the resultant?

Ans. 65 lb.

(383) If two forces of 27 lb. and 46 lb., respectively, act in exactly opposite directions upon a body, what is the resultant?

(384) The entire solar system is moving through space at the rate of 18 miles per second; (a) what is its velocity in miles per hour? (b) How far will it go in one day?

(385) Two bodies, starting from the same point, move in opposite directions, one at the rate of 11 ft. per second, and the other at the rate of 15 miles per hour; (a) what will be the distance between them at the end of 8 minutes? (b) How long before they will be 825 ft. apart?

$$\text{Ans. } \begin{cases} (a) \text{ 3 miles.} \\ (b) \text{ 25 seconds.} \end{cases}$$

(386) If six forces act towards the center of gravity of a body at angles of 30° , 45° , 135° , 210° , 225° , and 300° , whose magnitudes are 75, 47, 61, 32, 53, and 98 pounds, respectively, what is the value of their resultant? Solve by method of polygon of forces.

Ans. $45\frac{1}{2}$ lb.

(387) Suppose that the velocity of a steamboat in still water is 10 miles per hour, and that it is placed in a river flowing 4 miles per hour; also, that a man is walking the deck from stern to bow at the rate of 3 miles per hour. (a) What is the velocity of the boat when headed up stream? (b) When headed down stream? (c) Of the man in each case?

(388) A peg in the wall is pulled by two strings, one with a force of 21 lb., at an angle to the vertical of 45° , and the other with a force of 28 lb., at an angle of 60° ; what is the value and direction of the resultant when the forces are on opposite sides of the vertical line? Use the method of parallelogram of forces.

Ans. 30.34 lb.

(389) A force of 87 lb. acts at an angle of 23° to the horizontal; what are its horizontal and vertical components? Find, first, by the method of triangle of forces, and, second, by trigonometry.

Ans. $\begin{cases} 80.084 \text{ lb.} \\ 33.994 \text{ lb.} \end{cases}$

(390) A weight of 325 pounds rests upon a smooth inclined plane, as shown in Fig. 119, Art. 884. If the angle of the plane is 15° , (a) what is the perpendicular pressure against it? (b) What force would it be necessary to exert parallel to the plane, to keep it from sliding downwards, there being no friction? Solve by trigonometry, and also by the method of the triangle of forces.

Ans. $\begin{cases} (a) 313.93 \text{ lb.} \\ (b) 84.12 \text{ lb.} \end{cases}$

(391) If the weight of a body is 125 lb., what is its mass?

(392) If the mass of a body is 53.7, what is its weight?

(393) (a) Is the mass of a body always the same? (b) If the mass of a body on the earth's surface is 25, what would be its mass at the center of the earth? (c) On the surface of the moon?

(394) A body on the earth's surface weighs 141 lb.; (a) at what point above the surface will it weigh 100 lb.? (b) At what point below the surface? Take the earth's radius as 4,000 miles.

Ans. $\begin{cases} (a) 749.736 \text{ miles.} \\ (b) 1,163.12 \text{ miles.} \end{cases}$

(395) If a body were dropped from a balloon 1 mile above the earth's surface, (a) how long a time would it require to fall to the earth? (b) What would be its velocity when it struck?

Ans. $\begin{cases} (a) & 18.12 \text{ seconds.} \\ (b) & 582.76 \text{ ft. per sec.} \end{cases}$

(396) In the last example, if the body weighed 160 lb., what would be the kinetic energy on striking the earth?

Ans. 844,799 ft.-lb.

(397) If a cannon ball were fired vertically upward with a velocity of 2,360 ft. per second, (a) how high would it go? (b) How long would it take to fall to the earth?

Ans. $\begin{cases} (a) & 16.4 \text{ miles.} \\ (b) & 2 \text{ min. } 26.77 \text{ sec.} \end{cases}$

(398) The earth turns round once in 24 hours. If it were a perfect sphere 8,000 miles in diameter, how far would a point on its surface travel in one minute?

(399) If a projectile weighing 400 pounds be fired from a cannon with a velocity of 1,875 ft. per second, at a target 6 ft. distant, (a) what will be its kinetic energy, on striking the target, in foot-pounds? (b) In foot-tons? (c) If it penetrates but 6 in., what will be its striking force?

Ans. $\begin{cases} (a) & 21,863,339.55 \text{ ft.-lb.} \\ (b) & 10,931.67 \text{ ft.-tons.} \\ (c) & 43,726,679 \text{ lb.} \end{cases}$

(400) If the acceleration due to gravity were 20 ft. per second, instead of 32.16 ft. per second, how much longer would it take a body to fall to the earth from a height of 200 ft. than it does now?

Ans. 0.9454 second.

(401) What do you understand by center of gravity?

(402) What do you understand by specific gravity?

(403) (a) What is the density of a cubic foot of a body occupying a space of 800 cu. in., and weighing 500 lb.? (b) What is its specific gravity?

Ans. $\begin{cases} (a) & 33.582. \\ (b) & 17.28. \end{cases}$

(404) A body has been falling freely for 5 seconds what is its velocity?

(405) Suppose that a body, under certain conditions, were to fall freely for 3 seconds, and then fall uniformly with the velocity it had at the end of the third second for 6 seconds longer, how far would it fall? Ans. 723.6 ft.

(406) The weight of the head and piston of a steam hammer, together with the piston rod, is 8 tons. If it falls 8 ft. and compresses a mass of iron $\frac{1}{2}$ in., what is the force of the blow? Ans. 1,536 tons.

(407) Explain what you understand by centrifugal force.

(408) If a cast iron sphere 4 in. in diameter be revolved in a circle, in which the distance from the center of the sphere to the center of the circle is 15 in., what will be the tension of the string, the sphere making 60 revolutions per minute? Ans. 13.38 lb.

(409) The outside diameter of an engine fly-wheel is 80 in.; width of face, 26 in.; average thickness of rim, 5 in.; revolutions per minute, 175; what is the centrifugal force tending to burst the rim? Ans. 121,394+ lb.

(410) If a body weighs one pound at a distance of 100 miles from the center of the earth, (a) what will it weigh at the surface? (b) At 100 miles above the surface? Take the earth's radius as 4,000 miles.

Ans. $\left\{ \begin{array}{l} (a) \text{ 40 lb.} \\ (b) \text{ 38.072 lb.} \end{array} \right.$

(411) What would be the horsepower of a machine that could raise 10,746 lb. 354 ft. in 10 minutes?

Ans. 11.5275 H. P.

(412) How far above the surface of the earth will a 2-lb. ball weigh 3 oz.?

Ans. 9,064 miles.

(413) What is the range of a projectile thrown horizontally, 50 ft. above a level plain, the initial velocity being 140 ft. per second?

Ans. 246.87 ft.

(414) A projectile has an initial velocity of 30 ft. per second. How far below the horizontal line of direction will it strike a body 10 ft. away?

Ans. 1 ft. 9.44 in.

(415) What do you understand by moment of a force? Illustrate it.

- (416) Illustrate your idea of a couple.
- (417) Can a couple have a single resultant force?
- (418) Where is the center of gravity of a triangle located?
- (419) If $ABCD$ is a quadrilateral, and $AB = 10$ in., $BC = 8$ in., $CD = 7$ in., $DA = 9$ in., and the angle between AB and BC is 90° , where is the center of gravity of the figure? Determine it graphically.
- (420) Where is the center of gravity of a regular pentagon, the length of a side being seven inches?
- (421) If the weight of the balls shown in Fig. 128 were $W_1 = 21$ lb., $W_2 = 15$ lb., $W_3 = 17$ lb., and $W_4 = 9$ lb., where would the center of gravity be, the distance between the centers of W_1 and W_2 being 34 in. between W_2 and W_3 , 25 in.; between W_3 and W_4 , 40 in., and between W_4 and W_1 , 18 in.?
- (422) Find the center of gravity of a square board of uniform thickness whose sides are 14 in. in length, and having one of its corners cut off at a distance of 4 in., measured from that corner each way; or, what is the same thing, cutting off a right-angled triangle whose sides, including the right angle, are 4 in. long.
- (423) A bookbinder has a press, the screw of which has 4 threads to the inch. It is worked by a lever 15 in. long, to which is applied a force of 25 lb.; (a) what will be the pressure if the loss by friction is 5,000 lb.? (b) What would be the theoretical pressure? (c) What is the efficiency in this case?
- Ans. $\left\{ \begin{array}{l} (a) \text{ 4,424.8 lb.} \\ (b) \text{ 9,424.8 lb.} \\ (c) \text{ 46.95\%} \end{array} \right.$
- (424) How would you determine whether a body was stable or not if placed in a certain position?
- (425) If a prism 10 inches square and 22 inches long on its axis has been so cut that, when placed upon its base, its axis makes an angle of 60° with the horizontal, will it stand or fall? Consider the plane of the upper base as being at right angles to the axis.

(426) If the force moves through a distance of 5 ft. 6 in., while the weight is moving 6 in., (a) what is the velocity ratio of the machine? (b) What weight would a force of 5 lb. applied to the power arm raise?

(427) In the last example, if the efficiency were 65%, what weight could be raised?

(428) The length of a lever is 5 ft.; where must the fulcrum be placed, so that a weight of 35 lb. at one end may balance one of 180 lb. at the other end?

(429) In a block and tackle the theoretical force necessary to raise a weight of 1,000 lb. is 50 lb.; (a) what is the velocity ratio? (b) How many pulleys are there? (c) If the actual force necessary to raise the load is 95 lb., what is the efficiency?
Ans. (c) 52.63%.

(430) The nuts on a cylinder head are tightened by a wrench whose handle is 20 in. long. If the force exerted upon the wrench is 60 lb., and the efficiency of the combination is 40%, what pressure will the nut exert against the head (or, in other words, what is the tension of the stud), the pitch of the screw being $\frac{1}{8}$ of an inch? Ans. 24,127.5 lb.

(431) The base of an inclined plane is 20 ft. in length and its height is 5 ft.; (a) what force acting parallel to the plane will balance a weight of 1,580 lb.? (b) What force acting parallel to the base would balance this weight?

Ans. $\left\{ \begin{array}{l} (a) \text{ 383.2 lb.} \\ (b) \text{ 395 lb.} \end{array} \right.$

(432) Find what the weights W' and W'' must be to produce equilibrium when the levers shown in Fig. 5 are suspended from the ring A .

(433) If in Fig. 141, the power arms PF equal 14, 21 and 19 inches, respectively, and the weight arms

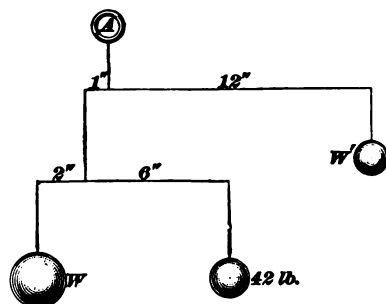


FIG. 5.

WF equal $2\frac{1}{2}$, $3\frac{1}{4}$ and $2\frac{3}{4}$ inches, respectively, what force applied at P will raise a load of 725 lb.? Ans. 3.032 lb.

(434) In Fig. 144, suppose the radius of the wheel A is 15 in.; of C , 12 in.; of E , 20 in.; of the drum F , 5 in.; of the pinion D , $3\frac{1}{2}$ in., and of B , 3 in.; (a) what load would a force of 35 lb. applied at P raise? (b) What is the velocity ratio? (c) If the weight actually raised was 1,932 lb., what is the efficiency? Ans. Efficiency = 80.5%.

(435) A frame having the shape of an equilateral triangle measuring 15 in. on each edge is suspended in a horizontal position, weights of 12, 15, and 18 lb., respectively, being hung from each corner. Where is the point of suspension that the frame may remain horizontal? Solve graphically, and measure the perpendicular distances from the point of suspension to each edge of the triangle.

(436) A stone weighing 500 lb. is balanced on the edge of the roof of a building 75 ft. high; (a) what is its potential energy? (b) If all of its potential energy could be changed into kinetic energy, without any loss through friction or heat, how many horsepower would be developed, on the supposition that the work was done in the same time that it would take the stone to fall freely to the ground? Ans. (b) 31.57 H.P.

(437) A cubic foot of a certain kind of stone weighs 127 lb.; what is its specific gravity? Ans. 2.032.

(438) The specific gravity of bismuth is 9.823; what is the weight of a cubic inch? Ans. .3553 lb.

(439) In the differential pulley shown in Fig. 149, the radius of the larger pulley is $6\frac{1}{2}$ in., and of the smaller pulley, $5\frac{3}{4}$ in.; what weight will a force P of 60 lb. raise if the efficiency of the mechanism is 48%? Ans. 499.2 lb.

(440) If a hammer whose head weighs $1\frac{1}{2}$ lb., strikes a nail with a velocity of 25 ft. per second, driving it $\frac{3}{8}$ of an inch into the wood, what is the force of the blow? Ans. 466.42 lb.

(441) If 4 cu. ft. of copper alloy weigh a ton (2,000 lb.), (a) what is its specific gravity? (b) What is the weight of a cu. in.? Ans. (a) Sp. Gr. 8.

(442) If the distance between the center line of the handle and the axis of the drum shown in Fig. 143 is $14\frac{1}{2}$ in., and the diameter of the drum is 5 in., what load will a force of 30 lb. exerted on the handle at P raise?

(443) If the coefficient of friction is .21, what force would be required to move a body weighing 75 lb.?

(444) If a man raises a weight of 900 lb. 150 feet in 15 minutes, by means of a fixed and movable pulley, (a) how much work has he done? (b) What part of a horsepower is this equivalent to?

Ans. (b) $\frac{3}{11}$ H.P.

(445) In the last example, what horsepower would the man have actually expended if the resistance due to friction had been 36% of the load?

Ans. .3709 H.P.

(446) If a force of 18 lb. is just sufficient to move a weight of 88 lb. along a horizontal plane, what is the coefficient of friction?

Ans. .2045.

(447) If 3 cu. ft. of a certain material weigh 1,200 lb., what is its density?

Ans. 12.438.

(448) An iron plate rests upon four supports; upon it is placed a weight of 125 lb.; a compressed spring, placed under the plate directly under the center of gravity of the plate and weight, exerts an upward pressure of $47\frac{1}{2}$ lb. What is the pressure upon each support?

(449) Find the resultant of the forces acting in Fig. 6—all acting towards the same point.

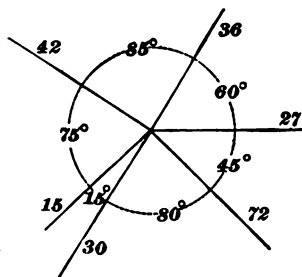


FIG. 6.

(450) The distance between the center line of the handle and the axis of the drum in Fig. 143 is 12 in., and the diameter of the drum is $4\frac{1}{2}$ in. The free end of the rope, forming part of a block and tackle having 6 pulleys, is wound up on this drum. How great a weight can be lifted by the pulleys if a force of 30 lb. is exerted on the handle?

Ans. 960 lb.

(451) (a) What is the velocity ratio in the last example?
(b) If the weight actually lifted by a force of 30 lb. was 790 lb., what is the efficiency?

Ans. $\left\{ \begin{array}{l} (a) \ 32. \\ (b) \ 82.29\%. \end{array} \right.$

(452) It is desired to raise a weight by means of a pulley fixed overhead, the free end of the rope passing over another pulley, fixed to the floor. If the resistance due to friction is 24% of the load lifted, (a) what force would be necessary to raise a weight of 475 lb.? (b) What is the efficiency?

Ans. $\left\{ \begin{array}{l} (a) \ 589 \text{ lb.} \\ (b) \ 80.64\%. \end{array} \right.$

(453) Explain your idea of work, power, horsepower, and kinetic energy. If a constant force of 6 pounds can cause a body weighing 60 pounds to move a distance of 25 feet in $2\frac{1}{2}$ seconds, what is (a) the work done? (b) The power expended?







